The Yoneda embedding in simplicial type theory

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Riehl and Shulman [1] introduced *simplicial type theory* (STT), a variant of homotopy type theory which aimed to study not just homotopy theory, but its fusion with category theory: $(\infty, 1)$-category theory. While notoriously technical, manipulating ∞-categories in simplicial type theory is often easier than working with ordinary categories, with the type theory handling infinite stacks of coherences in the background. We capitalize on recent work by Gratzer et al. [2] defining the $(\infty, 1)$-category of ∞-groupoids in STT to define presheaf categories within STT and systematically develop their theory. In particular, we construct the Yoneda embedding, prove the universal property of presheaf categories, refine the theory of adjunctions in STT, introduce the theory of Kan extensions, and prove Quillen's Theorem A. In addition to a large amount of category theory in STT, we offer substantial evidence that STT can be used to produce difficult results in ∞-category theory at a fraction of the complexity.

Dedicated to the dear memory of Thomas Streicher

1 Introduction

Russell [3] famously described two styles of formalizing mathematics as the difference between *theft* and *honest toil*. Both approaches can be seen in the present use of dependent type theory. Honest toil involves proceeding *analytically*: treating types as basic objects equivalent to sets and defining and reasoning about objects like the real numbers, groups, and topological spaces as one would ordinarily. This is what is done in e.g., the Coq proof of the Odd Order Theorem [4]. The more expeditious route of theft involves treating type theory as a bespoke *synthetic* language for a particular kind of mathematical object and postulating their basic properties. This narrows the scope of type theory but, by the same token, makes proofs about those particular objects far more concise. For instance, *homotopy type theory* (HoTT) [5] postulates various axioms that ensure that types behave like spaces (up to homotopy), making it possible to prove theorems from algebraic topology without ever introducing an explicit description of a space. In reality, the synthetic approach is less akin to theft than a loan; one pays for the customized type theory with a semantic model that interprets types as the intended objects and validates the additional axioms.

In this work, we embrace the synthetic methodology to use type theory to study category theory. In particular, we add various axioms to homotopy type theory in order to construct a system where HoTT's slogan "all types are spaces and all functions are continuous" is replaced by "(some) types are (∞-)categories and all functions are functors".[1] This extension of type theory is called *simplicial* type theory (STT) and was introduced by Riehl and Shulman [1].

While knowledge of ∞-categories is not necessary to use our theory, rough intuition for them is helpful for understanding STT. We therefore recall the following fuzzy definition. An ∞-category C is a collection of objects with a *space* of arrows between objects c and d, $\hom(c, d)$, rather than a set, equipped with a continuous composition operation and assignment of identity arrows. Crucially, the composition operation need only be associative and unital up to homotopy, but with the constraint

[1] In this paper, by ∞-category we mean $(\infty, 1)$-category.

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that those homotopies themselves satisfy *coherence* laws in the form of additional homotopies, and so on with coherences between coherences, etc. As a loose analogy, just as a monoidal category relaxes monoids by allowing $\otimes$ to be associative up to isomorphisms satisfying certain coherence equations, ∞-categories weaken ordinary categories to allow for the category laws to only hold up to (infinitely coherent) isomorphisms.

Remarkably, essentially every theorem one might hope for of ordinary categories holds for ∞-categories. However, the proofs are vastly more complex as they work with *models* of ∞-categories (tools used to organize and manage the tower of coherences [6]). The goal of STT is to use type theory to hide coherences from the user and to allow for proofs that are no more difficult than the classical arguments for 1-categories.

In this work, we provide substantial evidence of this hypothesis by developing a large swathe of category theory—several of the main results of *Categories for the Working Mathematician* [7]—purely within STT.

1.1 Simplicial type theory

To construct a type theory for synthetic category theory, one may hope to interpret type theory into the category of categories (∞ or otherwise) to ensure that types realize categories. However, the category Cat of small categories is too poorly behaved to form a model of Martin-Löf type theory (MLTT). Instead, Riehl and Shulman [1] enlarge Cat and embeds it as a reflective subcategory in the (∞-)presheaf category on the simplex category $\widehat{\Delta}$ which is rich enough to model HoTT. STT then axiomatizes some of $\widehat{\Delta}$ to isolate Cat as a reflective subuniverse within the type theory [8].

We will introduce the full suite of additions in Section 2 (collected in Appendix B for convenience), but the most important among them is the postulated *interval type* $\mathbb{I} : \mathcal{U}_0$. We further assume that $\mathbb{I}$ is a bounded linear order with endpoints $0, 1 : \mathbb{I}$. Intuitively, $\mathbb{I}$ is meant to capture the category $\{0 \to 1\}$—it is interpreted as such in $\widehat{\Delta}$—and we may use this to define and probe the type of *synthetic morphisms* in an arbitrary type X: an arrow in X corresponds to an ordinary function $\mathbb{I} \to X$ with evaluation at $0, 1$ yielding the domain and codomain. For instance, the identity arrow at $x : X$ is given by $\lambda\_.\, x$.

However, just as the intended model $\widehat{\Delta}$ is strictly larger than Cat, not all types in STT faithfully model categories. In particular, while one is always able to construct identity morphisms, not all types enjoy a composition operator. Remarkably, however, composition operators are unique when they exist and their existence for a type X is captured by a relatively short proposition (Definition 2.8). With a composition operation for X to hand, we can define the type of isomorphisms in X and we define a category to be a type where (1) the composition operation exists uniquely up to homotopy, and (2) the type of isomorphisms in X is equivalent to the identity type $=_X$.

Remark 1.1. This last point hinges crucially on *not* assuming the uniqueness of identity proofs lest we accidentally forbid any synthetic category from having an object with a non-trivial automorphism. However, by assuming isomorphisms and identify proofs coincide, we are able to leverage type theory's support for replacing equals by equals to seamlessly transport proofs along isomorphisms. This is why working with HoTT/intensional type theory when formulating synthetic category theory proves more convenient than extensional type theory, even if one is unconcerned with ∞-categories.

1.2 Category theory inside of STT

While some recent work has investigated STT for its applications to programming languages [2, 9, 10], the majority of work on simplicial type theory has focused on proving results from category theory inside of type theory [1, 11–16]. To this end, the theory of adjunctions, discrete and Grothendieck fibrations, and (co)limits have been introduced and studied within simplicial type theory. Some of these results, e.g., a fibrational Yoneda lemma [1], were subsequently mechanized [17].

Until recently, however, there were no closed types in STT which represented non-trivial categories. As a result, while an excellent definition of adjunctions is presented by Riehl and Shulman [1], no examples can be given. This was changed by Gratzer et al. [2] who extended STT to construct the (∞-categorical version of) the category of sets $\mathcal{S}$. Objects of $\mathcal{S}$ are elements of $\mathcal{U}_0$ that encode ∞-groupoids and morphisms in $\mathcal{S}$ correspond to functions thereof. *Op. cit.* uses $\mathcal{S}$ as a building block to recover algebraic categories (groups, rings) as well as other examples (posets, the simplex category, etc.).

Gratzer et al. extends HoTT with various *modalities* to construct $\mathcal{S}$. While we take $\mathcal{S}$ wholesale, some modalities they used are critical for stating natural theorems in category theory. Accordingly, we will work within a modal extension of HoTT based on MTT [18] within this paper.

1.3 Contributions

We revisit the basic category theory in light of the construction of $\mathcal{S}$ and show that the majority of classical results one encounters in category theory are now within reach of simplicial type theory. For the first time, we show that STT can be used to prove vital theorems in ∞-category theory without recourse to complex models. Many of these theorems (e.g., fully-faithful essentially surjective functors are equivalences) do not explicitly mention $\mathcal{S}$, but crucially rely on the reasoning principles enabled by $\mathcal{S}$. We prove two workhorse results from presheaf categories $\widehat{C}$:

- We construct a fully-faithful function $\mathbf{y} : C \to \widehat{C}$.
- We prove that $\widehat{C}$ is the "free cocompletion of C".

The key technical innovation for these is the *twisted arrow category*, which we integrate into STT as a modality. We are then able to deduce various classical results, e.g.:

- that pointwise invertible maps in $C \to D$ are invertible;
- that pointwise left adjoints are left adjoints;
- that (co)limits are computed pointwise in $C \to D$;
- the theory and existence of pointwise Kan extensions;
- Quillen's theorem A;
- the properness of cocartesian fibrations.

The synthetic approach yields concise proofs for many of these theorems compared with classical expositions in 1-category theory, but our proofs apply to ∞-categories as well and there the improvements are far more radical: it takes hundreds of pages for Lurie [19] to prove that $\mathbf{y}$ is fully-faithful and the proof that pointwise natural transformations are isomorphisms takes nearly five pages of effort by Cisinski [20]. By dividing work between a construction within STT and the already-existing model of STT, we are able to avoid many of these technicalities and give proofs more familiar to 1-category theorists. In particular, we show that just as homotopy type theory allowed type theorists to produce new arguments in algebraic topology, simplicial type theory enables type theorists to do the same with ∞-category theory.

Remark 1.2. Given that STT extends HoTT with a number of axioms, it is natural to ask whether these axioms are *complete* in any sense. Our present suite of axioms is not complete for the intended models of simplicial objects in an ∞-topos (though they are sound) but this is neither surprising nor undesirable: HoTT itself is not complete for its intended models (∞-topoi) and its exotic models are a source of considerable interest. Similarly, we expect STT to have interesting exotic models and cannot reasonably hope for a finite set of axioms to be complete for standard models. What is far more important is whether these axioms suffice to derive the standard results in category theory, an empirical rather than a mathematical question. Indeed, in related synthetic approaches to domain theory [21], differential geometry [22], and algebraic geometry [23], the precise axioms arose over the course of multiple years and several iterations. To this end, we view our results as providing firm evidence towards the expressivity of this axiom set.

1.4 Organization

In Section 2 we review the highlights of the basis of this work: homotopy type theory, basic simplicial type theory, modal homotopy type theory, and their synthesis: STT. In Section 3, we study the *twisted arrow* category and use it to construct the Yoneda embedding. We prove several increasingly sophisticated versions of the Yoneda lemma and conclude with a fully functorial version (Theorem 3.12). In Section 4 we put the Yoneda lemma to work to revisit the theory of adjunctions given by Riehl and Shulman [1]. We develop several tools for constructing adjunctions and use them to give the first non-trivial examples of adjunctions in STT. We also use this machinery to show that $\widehat{C}$ is the free cocompletion of C (Theorem 4.16). In Section 5 we develop the theory of Kan extensions in STT and prove several vital results: the existence of pointwise Kan extensions (Theorem 5.3), Quillen's theorem A (Theorem 5.11), and the properness of cocartesian maps (Theorem 5.20). Our proof of the last fact is particularly notable, as our use of type theory led us to a far simpler proof than those we are aware of in the literature.

Remark 1.3. For reasons of space, we have relegated the formal rules of our type theory to Appendix A and details of selected results (those marked with $*$) to Appendix C.

2 Modal and simplicial type theory

In this paper we take STT largely for granted and focus on working within the theory. However, to make this paper more self-contained, we devote this section to carefully explaining the novel constructs of modal homotopy type theory and the axioms supplementing it which form simplicial type theory.

2.1 Homotopy type theory

We begin by recalling the basic concepts and notation from homotopy type theory we use in this paper. The canonical reference is the HoTT book [5]. We work within intensional Martin-Löf type theory and note how HoTT extends this.

Notation 2.1. We write $a =_A b$ for the identity type (often suppressing A). Given $p : a =_A b$ and $B : A \to \mathcal{U}$, we write $p_!$ for the map $B(a) \to B(b)$.

Definition 2.2. We say that a function $f : A \to B$ is an equivalence if f admits both a left and a right inverse:

$$\mathsf{isEquiv}(f) = \textstyle\sum_{g,h:B\to A}(g \circ f = \mathsf{id}) \times (f \circ h = \mathsf{id})$$

We write $A \simeq B$ for the sum $\sum_{f:A\to B} \mathsf{isEquiv}(f)$.

HoTT is an extension of intensional type theory with a hierarchy of universes satisfying the univalence axiom:

$$\mathsf{univ}_i : \textstyle\prod_{A,B:\mathcal{U}_i} \mathsf{isEquiv}(\lambda p.\,(p_!, \cdots) : A =_{\mathcal{U}_i} B \to A \simeq B)$$

We shall suppress the i in univ and $\mathcal{U}$ and ignore size issues unless they are relevant. Univalence produces a great number of paths in $\mathcal{U}$ that are distinct from refl. We are often interested in types that are trivial, have only trivial paths, or trivial paths between paths, etc. These conditions are organized into a family of predicates referred to as the truncation level $(-2, -1, 0, \ldots)$ of a type. We will only use the first three levels, stating that a type is contractible or a (homotopy) proposition or set:

$$\mathsf{isContr}(A) = \sum_{a:A} \prod_{b:A} a = b \qquad \mathsf{isProp}(A) = \prod_{a,b:A} \mathsf{isContr}(a = b)$$

$$\mathsf{isSet}(A) = \prod_{a,b:A} \mathsf{isProp}(a = b)$$

Proposition 2.3 (Shulman [24]). *All type-theoretic model topoi (and, therefore, Grothendieck ∞-topoi) model* HoTT.

We shall also have occasion to use various *higher inductives types* (HITs). The semantics of HITs is complex and not directly addressed by the above result [25]. In particular, while Shulman [24] shows that the above model supports all higher inductive types, he does not show that universes are strictly closed under these constructions. While it is work-in-progress to obtain this result, it is easy to show that universes are *weakly* closed under these constructions. For instance, there exists a type $D : \mathcal{U}_0$ such that $D \simeq A \amalg_C B$ whenever $A, B, C : \mathcal{U}_0$. Accordingly, we shall assume that our universes are closed under higher inductive types, albeit only with propositional β rules.

2.2 Simplicial type theory

With HoTT to hand, we turn to simplicial type theory. This is an extension of HoTT by a handful of axioms that allow us to treat (certain) types as $(\infty, 1)$-categories, henceforth just referred to as categories. We will consequently drop the $(\infty, 1)$- or ∞-prefix everywhere. First and most fundamentally, we add the following:

Axiom A. *There is a set $\mathbb{I}$ that forms a bounded distributive lattice $(0, 1, \vee, \wedge)$ such that $\prod_{i,j:\mathbb{I}} i \leq j \vee j \leq i$ holds.*

We view $\mathbb{I}$ as a *directed interval*, and Riehl and Shulman [1] use this to equip every type with a notion of synthetic morphism:

Definition 2.4. A synthetic morphism $f : \hom_X(x, y)$ where $x, y : X$ is a function $f : \mathbb{I} \to X$ together with propositional equalities $f\,0 =_X x$ and $f\,1 =_X y$.

One can define the identity morphism $\mathsf{id}_x : x \to x$ as $\lambda\_.\, x$. Moreover, every function $f : X \to Y$ automatically has an action on synthetic morphisms $\alpha : \mathbb{I} \to X$ by post-composition $f \circ \alpha : \mathbb{I} \to Y$. In this case, we often write $f(\alpha)$.

From $\mathbb{I}$ we immediately obtain the n-cubes $\mathbb{I}^n$ and from them we can isolate simplices Δ^n, boundaries $\partial\Delta^n$, and horns Λ^n_k. In particular, $\Delta^2 \to X$ represents an 2-cell in X witnessing the composite of two arrows, and $\Lambda^2_1 \to X$ represents a pair of composable arrow (without a composite). We recall the definitions of these types below:

$$\Delta^n = \{(i_1, \ldots, i_n) : \mathbb{I}^n \mid i_1 \geq i_2 \geq \cdots \geq i_n\} \qquad \Lambda^2_1 = \{(i, j) : \mathbb{I}^2 \mid i = 1 \vee j = 0\}$$

Notation 2.5. We write $i : \Delta^n$ $(0 \leq i \leq n)$ as shorthand for sequence of i copies of 1 followed by 0: $(1, 1, \ldots, 0, \ldots)$.

A map $f : \Delta^2 \to X$ is said to witness that the composite of $f(-, 0)$ followed by $f(1, -)$ is $\lambda i.\, f(i, i)$. We emphasize that this is *data*; there can be many distinct f's witnessing the same composition as X may have many non-equivalent 2-cells with the same boundary. By the same token however, it is not always the case that a pair of composable morphisms $\Lambda^2_1 \to X$ extends to a composition datum $\Delta^2 \to X$. This is precisely because not every type in STT can be regarded as a category; even though we have defined $\hom_X(x, y)$ for every X, there is no *a priori* way of *composing* these morphisms. Precategories are types for which all composites exist:

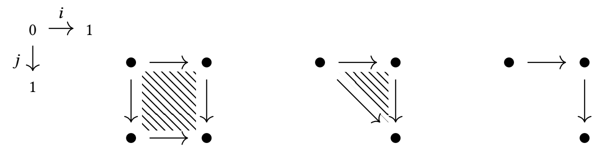

Fig. 1. Visualization of $\mathbb{I}^2$, Δ^2, and Λ^2_1.

Definition 2.6. A *precategory* is a type X satisfying the Segal condition: $\mathsf{isEquiv}(X^{\Delta^2} \to X^{\Lambda^2_1})$.

Roughly, the Segal condition ensures that every pair of composable morphisms in X extends (uniquely) to a 2-cell witnessing their composition and, in particular, there is an induced composition function $\hom(x, y) \times \hom(y, z) \to \hom(x, z)$. Uniqueness automatically ensures that this operation is associative and unital. The definition of a category refines this slightly. In a precategory X we are able to define the type of isomorphisms $x \cong y$ between $x, y : X$ and so there are two potentially distinct types of evidence for x and y being identical: $x =_X y$ and $x \cong_X y$. A category is a precategory for which these two types are canonical equivalent.

Definition 2.7. $\alpha : \hom(x, y)$ is an isomorphism ($\mathsf{isIso}(\alpha)$) if there exist $\beta_0, \beta_1 : \hom(y, x)$ such that $\beta_0 \circ f = \mathsf{id}, f \circ \beta_1 = \mathsf{id}$.[2] We write $\mathrm{iso}(x, y)$ or $x \cong y$ for the subtype of isomorphisms.

Definition 2.8. A precategory C is a *category* if it satisfies the Rezk condition:

$$\prod_{x,y:C} \mathsf{isEquiv}((x = y) \to \mathrm{iso}(x, y))$$

If every morphism in C is an isomorphism, then C is a *groupoid*.

Example 2.9. $\mathbb{I}, \Delta^n, \mathbb{I}^n$ are all categories [2].

Lemma 2.10. *C is a groupoid if and only if* $\mathsf{isEquiv}(C \to C^{\mathbb{I}})$ (C *is* $\mathbb{I}$-null [8]).

Riehl and Shulman [1] develop the basic theory of these synthetic categories. As noted above, every function has an action on morphisms and *op. cit.* shows that this action preserves compositions and identities and therefore defines a functor. They also show that $C \to D$ is then a category whenever D is, and that synthetic morphisms $\hom_{D^C}(f, g)$ are precisely natural transformations. One can reformulate various classical categorical notions rather directly:

Definition 2.11 (Bardomiano Martínez [12]). A natural transformation $\alpha : \hom_{C^I}(\mathrm{const}(c), F)$ witnesses c as the limit of $F : C^I$ if α induces an equivalence $\hom(c', c) \simeq \hom(\mathrm{const}(c'), F)$.

Definition 2.12. An adjunction between two categories C, D consists of a pair of functions $f : C \to D$ and $g : D \to C$ with a natural isomorphism $\iota : \prod_{c,d} \hom(f(c), d) \simeq \hom(c, g(d))$.

Definition 2.13. A category C is i-(co)complete if for every category $D :_\flat \mathcal{U}_i$, $\mathsf{const} : C \to C^D$ is a (right) left adjoint.

While we have given a few examples of categories above, a notable type that is *not* category is the universe $\mathcal{U}$. Maps $A : \mathbb{I} \to \mathcal{U}$ are too unstructured to compose and, in particular, correspond neither to functions $A(0) \to A(1)$ nor $A(1) \to A(0)$ (consider $\lambda i.\, i = 0$ or $\lambda i.\, i = 1$). In Section 2.4, we shall discuss the subuniverse $\mathcal{S}$ constructed by Gratzer et al. [2], which is a category of groupoids whose morphisms correspond to functions. To properly situate this definition, we recall what it means for $X : A \to \mathcal{U}$ to be covariant [1], giving an assignment from morphisms $\hom(a_0, a_1)$ to functions $X(a_0) \to X(a_1)$.

[2]This is precisely the HoTT equivalence but recast into synthetic morphisms.

Definition 2.14. A family $X : A \to \mathcal{U}$ is *covariant* if for every $a : \hom(a_0, a_1)$ and $x_0 : X(a_0)$, the following is contractible:

$$\mathsf{Lift}(a, x_0) = \\ \textstyle\sum_{x_1:X(a_1)} \sum_{x:\hom((a_0,x_0),(a_1,x_1))} \pi_1(x) =_{\hom(a_0,a_1)} a$$

Here, x is a morphism in $\sum_{a:A} X(a)$.

Since $\mathsf{Lift}(a, x_0)$ is contractible it has an inhabitant x_1. This yields a function $a, x_0 \mapsto x_1$ which defines $a_! : X(a_0) \to X(a_1)$. The contractibility of $\mathsf{Lift}(a, x_0)$ ensures that these functions compose correctly, etc.

Notation 2.15. Given $X : A \to \mathcal{U}$ we write $\widetilde{X}$ for $\sum_{a:A} X(a)$. We further say the projection $\sum_{a:A} X(a) \to A$ is covariant when X is. For a general map $\pi : X \to A$ we write X_a for $\sum_{x:X} \pi(x) = a$ and say π is covariant when $\lambda a.\, X_a$ is.

Lemma 2.16. *A family $X : A \to \mathcal{U}$ is covariant if and only if the map $\lambda p.\, (p(0), \pi_1 \circ p) : \widetilde{X}^{\mathbb{I}} \to \widetilde{X} \times_A A^{\mathbb{I}}$ is an equivalence.*

In Sections 4.1 and 5.3, we shall briefly use a weakening of covariance:

Definition 2.17. A map $\pi : D \to C$ is *cartesian* if $\bar{\pi} : D^{\mathbb{I}} \to C^{\mathbb{I}} \times_{C^{\{1\}}} D^{\{1\}}$ is a right adjoint $\ell \dashv \bar{\pi}$ such that $\bar{\pi} \circ \ell = \mathsf{id}$.

One can give an equivalent characterization in terms of cartesian morphisms and show that e.g., every morphism in D can be factored as a vertical morphism followed by a cartesian morphism, see Buchholtz and Weinberger [14]

Finally, we note that since categories and groupoids are defined by certain *orthogonality* conditions, by Rijke et al. [8] they define reflective subuniverses.

Proposition 2.18. *There are idempotent monads $\bigcirc_{\mathsf{cat}}, \bigcirc_{\mathsf{grpd}}$ such that, e.g., $\bigcirc_{\mathsf{cat}} X$ is a category and $C^{\bigcirc_{\mathsf{cat}} X} \simeq C^X$ when C is a category.*

Proposition 2.19 (Riehl and Shulman [1])**.** *When Proposition 2.3 is specialized to simplicial spaces ($\widehat{\Delta}$), the resulting model validates Axiom A and in this model categories are realized by ∞-categories (modeled by complete Segal spaces) and groupoids by ∞-groupoids.*

2.3 Modal homotopy type theory

Many theorems in category theory require the ability to quantify over the objects in a category, e.g., "if $\alpha : F \to G$ is a natural transformation of functors $C \to \mathcal{D}$ and each α_c is invertible, then α is invertible". A version of this is proven by Riehl and Shulman [1]: $\left(\prod_{c:C} \mathsf{isIso}(\lambda i.\, \alpha\, i\, c)\right) \to \mathsf{isIso}(\alpha)$, but this is subtly different as we discuss below. In fact, as it stands we cannot directly capture the classical statement in STT.

To understand the divergence between the STT and classical results, note that by working internally to type theory when proving $\prod_{c:C} \mathsf{isIso}(\lambda i.\, \alpha\, i\, c)$ we cannot assume that c is just an object in C: since it is an arbitrary element, we have to assume it is constructed in an arbitrary context which might contain, e.g., a copy of $\mathbb{I}$ such that c represents a synthetic morphism. In fact, if we unfold the above type into the model we find that constructing $\prod_{c:C} \mathsf{isIso}(\lambda i.\, \alpha\, i\, c)$ already entails proving that the chosen inverses are natural. A great deal of the power of simplicial type theory comes from this implicit naturality, but it makes this particular result weaker. After all, its purpose in standard category theory was that in this particular situation, *a priori* unnatural choices of inverses will automatically be natural. Moreover, we shall encounter theorems that are simply false when naively translated in this way.

Accordingly, to make STT practical we must extend it with *modalities*: unary type constructors distinguished by their failure to respect substitution or apply in arbitrary contexts. For instance, we shall eventually equip STT with a modality $\langle \flat \mid - \rangle$ which discards all non-invertible synthetic morphisms from a type to produce its *core*, which we then use to faithfully encode pointwise invertibility (see Example 2.22).

A complete reference to the modal type theory we use—MTT [18]—is given by Gratzer [26] and we record formal rules in Appendix A. Fortunately, the rules for, e.g., $\sum$-types are unaffected by the addition of modalities, so for reasons of space we only recall the new rules which must be added to MLTT to extend HoTT with modalities à la MTT.

MTT is parameterized by a *mode theory*: a strict 2-category describing the collection of modalities (the morphisms) available along with the natural transformations between them (the 2-cells). We use μ, ν, ξ to range over modalities. In the case of simplicial type theory, our mode theory will have only one object along with a handful of generating modalities and 2-cells. In fact, the 2-category is locally posetal: there is at most one 2-cell between any pair of 1-cells. We therefore specify the mode theory as the partially-ordered monoid generated by $\flat, \sharp, \mathsf{op}, \mathsf{tw}$ and subject to the following (in)equalities:

$$\flat \circ \flat = \flat \circ \mathsf{op} = \flat \circ \sharp = \flat \qquad \sharp \circ \sharp = \sharp \circ \mathsf{op} = \sharp \circ \flat = \sharp \qquad \mathsf{op} \circ \mathsf{op} = \mathsf{id} \qquad \flat \leq \mathsf{id} \leq \sharp \qquad \flat \leq \mathsf{tw}$$

Each morphism μ in the mode theory induces a modal type $\langle \mu \mid - \rangle$. We will describe the rules for these modal types in a moment, but first we give some idea of what they are intended to denote. For now this is merely intuition, though the axioms and model described in Section 2.4 will make it so. As already mentioned, $\langle \flat \mid - \rangle$ removes all non-identity synthetic morphisms from a type, and $\langle \sharp \mid - \rangle$ is the right adjoint to this operation. Next, $\langle \mathsf{op} \mid - \rangle$ sends a type to its opposite and, in particular, reverses the directions of all synthetic morphisms. Finally, $\langle \mathsf{tw} \mid - \rangle$ sends a type to its corresponding type of *twisted arrows*; we shall analyze it in more depth in Section 3.

The formation rule for $\langle \mu \mid - \rangle$ is complex: the entire point of modalities is that $\Gamma \vdash A$ does not imply $\Gamma \vdash \langle \mu \mid A \rangle$. Instead, MTT introduces a novel form of context operation which acts like a "left adjoint" to $\langle \mu \mid - \rangle$:

$$\frac{\vdash \Gamma}{\vdash \Gamma, \{\mu\}} \qquad \frac{\Gamma, \{\mu\} \vdash A}{\Gamma \vdash \langle \mu \mid A \rangle} \qquad \frac{\Gamma, \{\mu\} \vdash a : A}{\Gamma \vdash \mathsf{mod}_{\mu}(a) : \langle \mu \mid A \rangle}$$

We refer to $\{\mu\}$ as a *modal restriction*. It is helpful to compare $\langle \mu \mid A \rangle$ with dependent products and, therefore, to see $-, \{\mu\}$ as extending the context by something akin to a substructural "μ variable" [27, 28]. The real force of modalities comes through how these $\{\mu\}$s interact with variables. In particular, it is not the case that $\Gamma, x :A, \{\mu\} \vdash x : A$; since $-, \{\mu\}$ is intended to model a left adjoint, we cannot generally assume that there is a weakening substitution $\Gamma, \{\mu\} \to \Gamma$. Instead, we alter the rule extending a context with a variable so that each variable is annotated with a modality:

$$\frac{\vdash \Gamma \qquad \Gamma, \{\mu\} \vdash A}{\vdash \Gamma, x :_{\mu} A} \qquad \frac{\mu \leq \mathsf{mods}(\Gamma_1)}{\Gamma_0, x :_{\mu} A, \Gamma_1 \vdash x : A}$$

The original context extension is given by taking $\mu = \mathsf{id}$. In the second rule, $\mathsf{mods}(\Gamma_1)$ is the composite $\nu_0 \circ \nu_1 \circ \cdots$ of all the $\{\nu_i\}$s occurring in Γ_1 (and is id if there are no such occurrences). In other words, a variable with annotation μ can be used precisely when it occurs behind a series of modal restrictions which total to something greater than or equal to μ. It is therefore in the variable rule where the partial ordering on modalities comes into play.

Lemma 2.20. *If* $\Gamma, x :_{\mu} A \vdash B$, $\Gamma, x :_{\mu} A \vdash b : B$, *and* $\Gamma, \{\mu\} \vdash a : A$, *then* $\Gamma \vdash B[a/x]$ *and* $\Gamma \vdash b[a/x] : B[a/x]$.

The final piece of the puzzle is the elimination rule for modalities. Roughly, this rule says that modal annotations are equivalent to modal types "from the perspective of a type", i.e., that giving an element in context $\Gamma, x :_\nu \langle \mu \mid A \rangle$ is the same as giving one in $\Gamma, x :_{\nu \circ \mu} A$. This concretely amounts to the following pattern-matching rule which allows us to assume that $x :_\nu \langle \mu \mid A \rangle$ is of the form $\mathsf{mod}_\mu(y)$ where $y :_{\nu\circ\mu} A$:

$$\frac{\Gamma, x :_\nu \langle \mu \mid A \rangle \vdash B \qquad \Gamma, y :_{\nu\circ\mu} A \vdash b : B[\mathsf{mod}_\mu(y)/x] \qquad \Gamma, \{\nu\} \vdash a : \langle \mu \mid A \rangle}{\Gamma \vdash \mathsf{let}\ \mathsf{mod}_\mu(y) \leftarrow a\ \mathsf{in}\ b : B[a/x]}$$

$$\frac{\Gamma, x :_\nu \langle \mu \mid A \rangle \vdash B \qquad \Gamma, y :_{\nu\circ\mu} A \vdash b : B[\mathsf{mod}_\mu(y)/x] \qquad \Gamma, \{\nu \circ \mu\} \vdash a : A}{\Gamma \vdash (\mathsf{let}\ \mathsf{mod}_\mu(y) \leftarrow \mathsf{mod}_\mu(a)\ \mathsf{in}\ b) = b[a/y] : B[\mathsf{mod}_\mu(a)/x]}$$

While these rules account for all of the necessary extensions to handle modal types, we avail ourselves of a convenience feature as well, modal $\prod$-types:

$$\frac{\Gamma, x :_\mu A \vdash b : B}{\Gamma \vdash \lambda x.b : \prod_{x:_\mu A} B} \qquad\qquad \frac{\Gamma \vdash f : \prod_{x:_\mu A} B \quad \Gamma, \{\mu\} \vdash a : A}{\Gamma \vdash f(a) : B[a/x]}$$

Notation 2.21. "If $c :_\flat C$, then $\Phi(c)$" signifies $\prod_{c:_\flat C} \Phi(c)$.

Example 2.22. A faithful translation of "pointwise invertibility implies invertibility" where $C, D :_\flat \mathcal{U}$ and $\alpha : C \times \mathbb{I} \to D$ is $\left(\prod_{c:_\flat C} \mathsf{isIso}(\lambda i.\, \alpha\, i\, c)\right) \to \mathsf{isIso}(\alpha)$

Immediately from these rules, we may prove the following:

Proposition 2.23 (Gratzer et al. [18])**.**

- $\langle \mu \mid - \rangle$ *commutes with* $\sum$ *and* $\mathbf{1}$
- $\langle \mu \mid \langle \nu \mid - \rangle \rangle \simeq \langle \mu \circ \nu \mid - \rangle$ *and* $\langle \mathsf{id} \mid - \rangle \simeq \mathsf{id}$
- *If* $\mu \le \nu$*, then there is a map* $\mathsf{coe}^{\mu\le\nu} : \langle \mu \mid - \rangle \to \langle \nu \mid - \rangle$.
- $\langle \flat \mid \langle \flat \mid A \rangle \to B \rangle \simeq \langle \flat \mid A \to \langle \sharp \mid B \rangle \rangle$
- $\langle \flat \mid \langle \mathsf{op} \mid A \rangle \to B \rangle \simeq \langle \flat \mid A \to \langle \mathsf{op} \mid B \rangle \rangle$

The first point yields $\circledast : \langle \mu \mid A \to B \rangle \to \langle \mu \mid A \rangle \to \langle \mu \mid B \rangle$.

When it is not ambiguous, we will also occasionally suppress the equivalences $\langle \mu \mid \langle \nu \mid A \rangle \rangle \simeq \langle \mu \circ \nu \mid A \rangle$ and $\langle \mathsf{id} \mid A \rangle \simeq A$.

Notation 2.24. If $\Gamma, \{\mu\} \vdash f : A \to B$, we write $f^\dagger$ for the function $\mathsf{mod}_\mu(f) \circledast -$.

In general, $\langle \mu \mid - \rangle$ need not commute with propositional equality. However, this is true in our intended models and so we impose it as an axiom:

Axiom B. *The map* $\mathsf{mod}_\mu(a) = \mathsf{mod}_\mu(b) \to \langle \mu \mid a = b \rangle$ *sending* refl *to* $\mathsf{mod}_\mu(\mathsf{refl})$ *is an equivalence for all* $a, b :_\mu A$.[3]

Corollary 2.25. *Each* $\langle \mu \mid - \rangle$ *commutes with* (HoTT) *pullbacks* $A \times_C B = \sum_{a:A} \sum_{b:B} f(a) =_C g(b)$.

Remark 2.26. For readers familiar with *spatial type theory* [30], this modal type theory is an extension of spatial type theory to include two additional modalities (op, tw). In particular, the results of Shulman [30] that deal with $\flat$ and $\sharp$ can be reproduced in this setting.

[3]Technically, this map is defined by path induction in the family $\lambda(x, y : \langle \mu \mid A \rangle).\ \mathsf{let}\ \mathsf{mod}_\mu(a) \leftarrow x\ \mathsf{in}\ \mathsf{let}\ \mathsf{mod}_\mu(b) \leftarrow y\ \mathsf{in}\ \langle \mu \mid a = b \rangle$. By Gratzer [29], there is a computational account of this principle.

2.4 Modalities and simplicial type theory

To connect the modal and simplicial structures, we impose the following axioms motivated by the intended model, as described in Proposition 2.19 (and more generally $\mathcal{E}^{\Delta^{\mathrm{op}}}$ for an ∞-topos $\mathcal{E}$); see also the work of Myers and Riley [31]. First, the opposite map should be an anti-equivalence of $\mathbb{I}$:

Axiom C. *There is an equivalence* $\neg : \langle \mathsf{op} \mid \mathbb{I} \rangle \to \mathbb{I}$ *which swaps* 0 *for* 1 *and* $\vee$ *for* $\wedge$.

Corollary 2.27. $\langle \mathsf{op} \mid \Delta^n \rangle \simeq \Delta^n$.

Next, we require the two possible notions of discreteness (being $\mathbb{I}$-null or $\flat$-modal) to coincide:

Axiom D. *If* $A :_\flat \mathcal{U}$*, then* $\langle \flat \mid A \rangle \to A$ *is an equivalence* (discrete) *if and only if* $A \to A^{\mathbb{I}}$ *is an equivalence* ($\mathbb{I}$-null).

Axiom E. *The canonical map* $\mathsf{Bool} \to \mathbb{I}$ *is injective and induces an equivalence* $\mathsf{Bool} \simeq \langle \flat \mid \mathbb{I} \rangle$.

Motivated by our intended class of models, we insist that equivalences are jointly detected by Δ^n:

Axiom F. $f :_\flat A \to B$ *is an equivalence if and only if the following holds:*

$$\prod_{n :_\flat \mathsf{Nat}} \mathsf{isEquiv}((f_*)^\dagger : \langle \flat \mid \Delta^n \to A \rangle \to \langle \flat \mid \Delta^n \to B \rangle)$$

Note that since there is a section-retraction pair $\Delta^n \to \mathbb{I}^n \to \Delta^n$, we can replace Δ^n with $\mathbb{I}^n$ in the above principle.

One useful application is the following:

Lemma 2.28. *A map* $\pi :_\flat X \to A$ *is covariant if and only if the map* $\langle \flat \mid X^{\Delta^n} \rangle \to \langle \flat \mid X \rangle \times_{\langle \flat \mid A \rangle} \langle \flat \mid A^{\Delta^n} \rangle$ *induced by* $(0^*)^\dagger$ *and* $(\pi_*)^\dagger$ *is an equivalence for all* $n :_\flat \mathsf{Nat}$.

Finally, we add a new axiom to STT that governs tw. This axiom states that the n-simplices in $\langle \mathsf{tw} \mid X \rangle$ correspond to $(2n+1)$-simplices in X. For clarity, rather than directly stating this equivalence with Δ^{2n+1}, we use a slightly less general formulation relying on the *blunt join* $X \diamond Y$:

$$X \diamond Y = X \amalg_{X \times \{0\} \times Y} (X \times \mathbb{I} \times Y) \amalg_{X \times \{1\} \times Y} Y$$

This is the directed version of the join $X \star Y$ [5, Ch 6] such that $X \diamond Y$ is roughly $X \coprod Y$ with morphisms adjoined to connect each $x : X$ to each $y : Y$.

Lemma 2.29. *If* C *is a category, then* $C^{\Delta^{m+1+n}} \simeq C^{\Delta^m \diamond \Delta^n}$.

Lemma 2.30. $\langle \mathsf{op} \mid X \diamond Y \rangle \simeq \langle \mathsf{op} \mid Y \rangle \diamond \langle \mathsf{op} \mid X \rangle$.

Axiom G. *For every category* $C :_\flat \mathcal{U}$ *we have:*

- *Maps* $\pi_0^{\mathsf{tw}} : \langle \mathsf{tw} \mid C \rangle \to \langle \mathsf{op} \mid C \rangle$, $\pi_1^{\mathsf{tw}} : \langle \mathsf{tw} \mid C \rangle \to C$.
- *Equivalences* $\iota : \langle \flat \mid C^{\langle \mathsf{op} \mid \Delta^n \rangle \diamond \Delta^n} \rangle \simeq \langle \flat \mid \langle \mathsf{tw} \mid C \rangle^{\Delta^n} \rangle$ *and* $\tau : \langle \mathsf{tw} \mid C \rangle \simeq \langle \mathsf{tw} \mid \langle \mathsf{op} \mid C \rangle \rangle$.

We require that π_0^{tw}, π_1^{tw}, ι, *and* τ *be natural*[4] *and that the diagrams in* Fig. 2 *commute.*

One may visualize ι as ensuring that $\langle \flat \mid \Delta^n \to \langle \mathsf{tw} \mid C \rangle \rangle$ is isomorphic to a $2n+1$ simplex in C:

$$\begin{array}{ccccccc} c_n & \longleftarrow & c_{n-1} & \longleftarrow & \cdots & \longleftarrow & c_0 \\ \downarrow & & & & & & \\ c_{n+1} & \longrightarrow & c_{n+2} & \longrightarrow & \cdots & \longrightarrow & c_{2n} \end{array}$$

[4] By this, we mean that there is a choice of path filling for each naturality square, but we do not insist that these paths be coherent.

$$\begin{array}{ccc}
\langle \flat \mid C^{\langle \mathsf{op} \mid \Delta^n \rangle \diamond \Delta^n} \rangle & \longrightarrow & \langle \flat \mid C^{\langle \mathsf{op} \mid \Delta^n \rangle} \rangle \times \langle \flat \mid C^{\Delta^n} \rangle \\
\downarrow & & \downarrow \\
\langle \flat \mid \langle \mathsf{tw} \mid C \rangle^{\Delta^n} \rangle & \longrightarrow & \langle \flat \mid \langle \mathsf{op} \mid C \rangle^{\Delta^n} \rangle \times \langle \flat \mid C^{\Delta^n} \rangle
\end{array}$$

"Applying ι then $\pi_\epsilon^{\mathsf{tw}}$ picks out the appropriate restriction of $C^{\langle \mathsf{op} \mid \Delta^n \rangle \diamond \Delta^n}$."

$$\begin{array}{ccc}
\langle \flat \mid \langle \mathsf{op} \mid C \rangle^{\langle \mathsf{op} \mid \Delta^n \rangle \diamond \Delta^n} \rangle & \longrightarrow & \langle \flat \mid C^{\langle \mathsf{op} \mid \Delta^n \rangle \diamond \Delta^n} \rangle \\
\downarrow & & \downarrow \\
\langle \flat \mid \langle \mathsf{tw} \mid \langle \mathsf{op} \mid C \rangle \rangle^{\Delta^n} \rangle & \longrightarrow & \langle \flat \mid \langle \mathsf{tw} \mid C \rangle^{\Delta^n} \rangle
\end{array}$$

"Applying ι then τ reverses and swaps two sub-n-simplices."

$$\begin{array}{ccc}
\langle \mathsf{tw} \mid C \rangle & \longrightarrow & \langle \mathsf{tw} \mid \langle \mathsf{op} \mid C \rangle \rangle \\
\downarrow & & \downarrow \\
C \times \langle \mathsf{op} \mid C \rangle & \longrightarrow & \langle \mathsf{op} \mid C \rangle \times C
\end{array}$$

"τ swaps the actions of π_0^{tw} and π_1^{tw}."

The first two diagrams use Proposition 2.23 and the second Lemma 2.30.

Fig. 2. Laws for Axiom G.

The map π_1^{tw} picks out the bottom row and π_0^{tw} selects the top but *twisted* so that it lands in $\langle \mathsf{op} \mid C \rangle$ rather than C. This axiom will only be used in the proof of Theorem 3.4, where we use $\langle \mathsf{tw} \mid - \rangle$ to construct a bifunctorial version of hom.

Remark 2.31. Axiom G is not the most general axiom possible governing tw. For instance, it applies only to categories so that we may use Lemma 2.29 to make several maps in Fig. 2 more obvious. Moreover, we have required no coherences for either naturality equations or those diagrams in Fig. 2. We have found that these are not needed for Theorem 3.4 which is the main role of tw, so the extra complexity is not justified. We note, however, that it could be arranged for π_0^{tw}, π_1^{tw}, and τ to be induced by the mode theory (via coe) instead of postulating them. This has several advantages and in fact yields additional potentially useful equations. However, it would force us to consider a full 2-category for our mode theory rather than a locally posetal one. The addition of distinct 2-cells in the mode theory increases the complexity MTT slightly and since Axiom G is used once and modalities are used ubiquitously, we have opted for the above approach for ease of presentation.

Proposition 2.32 (Gratzer et al. [2])**.** *Proposition 2.19 extends to a model of modal* HoTT *validating our axioms.*

Remark 2.33. While Gratzer et al. [2] do not handle $\langle \mathsf{tw} \mid - \rangle$, the methods employed there scale directly to this situation. In particular, Mukherjee and Rasekh [32] give an explicit description of the necessary twisted arrow operation and shows it is a Quillen right adjoint as required to extend the model.

With modalities to hand, a number of results from classical category theory can be proven directly. For instance, the so-called fundamental theorem of ∞-category theory:

Theorem 2.34.* *If $C, D :_\flat \mathcal{U}$ are categories, then $F :_\flat C \longrightarrow D$ is an equivalence if (1) the induced map $\langle \flat \mid C \rangle \longrightarrow \langle \flat \mid D \rangle$ is surjective, and (2) for any $c, c' :_\flat C$ the map $\mathrm{hom}(c, c') \longrightarrow \mathrm{hom}(F(c), F(c'))$ is an equivalence.*

2.5 Basic building blocks for categories

Finally, we recall two results from Gratzer et al. [2] we shall use repeatedly within this work to construct new categories. The first is a construction of *full subcategories* using $\sharp$:

Proposition 2.35. *If* $C :_\flat \mathcal{U}$ *is a category and* $\phi :_\flat \langle \flat \mid C \rangle \to \mathsf{HProp}$*, then* $C_\phi = \sum_{c:C} \langle \sharp \mid \phi(\mathsf{mod}_\flat(c)) \rangle$ *is a category such that (1) the projection map* $C_\phi \to C$ *induces an equivalence on hom-types, (2)* $\langle \flat \mid C_\phi \rangle \simeq \sum_{c:\langle \flat \mid C \rangle} \phi(c)$*, and (3) a map* $F :_\flat D \to C$ *factors through* C_ϕ *if and only if* $\phi(\mathsf{mod}_\flat(F(d)))$ *holds for all* $d :_\flat D$*.*

Corollary 2.36. *If* $C, D :_\flat \mathcal{U}$ *are categories and* $F :_\flat C \to D$*, then* $\mathrm{hom}(c, c') \to \mathrm{hom}(F(c), F(c'))$ *is an equivalence for all* $c, c' : C$ *if and only if it an equivalence when* $c, c' :_\flat C$*.*

Next, we recall their construction of the category of groupoids which plays the role of the category of sets in simplicial type theory, e.g., we shall use this category to define presheaves:

Axiom/Proposition 2.37. *There is a category* $\mathcal{S}_i :_\flat \mathcal{U}_{i+1}$ *with an embedding* $\mathcal{S}_i \to \mathcal{U}_i$ *such that:*

- *If* $X : A \to \mathcal{S}_i$*, then the composite* $A \to \mathcal{U}_i$ *is covariant.*
- *The converse holds for* $A :_\flat \mathcal{U}_i$*,* $X :_\flat A \to \mathcal{U}_i$*: if* X *is covariant, then* X *factors through* $\mathcal{S}_i$*.*

Corollary 2.38 (Directed univalence)**.** $\mathcal{S}^{\mathbb{I}} \simeq \sum_{X_0, X_1 : \mathcal{S}} X_1^{X_0}$ *and composition in* $\mathcal{S}$ *is the composition of functions.*

Warning 2.39. Gratzer et al. [2] prove Axiom/Proposition 2.37 in a richer variation of STT (*triangulated type theory*). Since we only require this result, we take it as an "axiom" of sorts to work in a simpler type theory and note that one could extend STT to triangulated type theory to prove this theorem outright.

3 The Yoneda embedding

Within this section, we fix a category $C :_\flat \mathcal{U}$. Our goal is to study the type $\widehat{C} = \mathcal{S}^{\langle \mathrm{op} \mid C \rangle}$ of presheaves on C. As $\mathcal{S}$ is a category, so is $\widehat{C}$ and by directed univalence:

Lemma 3.1. $\mathrm{hom}(F, G) \simeq \prod_{c:\langle \mathrm{op} \mid C \rangle} F\,c \to G\,c$

Remark 3.2. Just as with e.g., completeness, $\widehat{C}$ implicitly fixes a universe level such that $\widehat{C} = \langle \mathrm{op} \mid C \rangle \to \mathcal{S}_i$. We may regard i as a parameter or simply take $i = 0$. Occasionally, we shall need to insist that $C \simeq C'$ where $C' : \mathcal{U}_i$ and in such situations we shall say that C is *small*. We assume all categories are locally small—that each $\mathrm{hom}_C(c, c')$ is small.

One may recast the *fibrational* Yoneda lemma proven by Riehl and Shulman [1] to take advantage of $\widehat{C}$ rather than quantifying over contravariant families as in *op. cit.*:

Lemma 3.3. $F(c) \cong \prod_{c':\langle \mathrm{op} \mid C \rangle} \mathrm{hom}_{\langle \mathrm{op} \mid C \rangle}(c, c') \to F(c')$

3.1 The twisted arrow category and the Yoneda embedding

In light of this last result, the natural next step is to define a map $C \to \widehat{C}$[5] which sends $c : C$ to something like $\mathrm{hom}(-, c)$. However, caution is required: $\mathrm{hom}(-, c)$ has type $C \to \mathcal{U}$ and not the required $\langle \mathrm{op} \mid C \rangle \to \mathcal{S}$. Upon reflection, the reader should find it surprising that $\mathrm{hom}(-, -) : C \times C \to \mathcal{U}$ at all; if all maps are functorial in STT how can $\mathrm{hom}(-, -)$ be covariant in both arguments? In fact, this is a consequence of the strange behavior of synthetic morphisms in $\mathcal{U}$.

[5] Here we see why C must be flat: we wish to discuss both C and $\langle \mathrm{op} \mid C \rangle$. It is helpful to understand $C :_\flat \mathcal{U}$ as a *closed* type which depends on nothing in the context and, in particular, need not be treated functorially.

While $\hom(-,-)$ is functorial in both arguments, the lack of directed univalence for $\mathcal{U}$ makes this useless. This strangeness ensures that $\hom(-,-)$ does not restrict to a function into $\mathcal{S}$.

What is required instead is a function $\Phi : \langle \mathsf{op} \mid C\rangle \times C \to \mathcal{S}$ such that $\Phi(\mathsf{mod}_{\mathsf{op}}(c), -) = \hom(c, -)$ whenever $c :_\flat C$, i.e., a function that agrees on objects with $\hom(-,-)$ and has the same functoriality in the second argument, but takes $\langle \mathsf{op} \mid C\rangle$ as its first argument. In fact, it is highly non-obvious where such a function should come from; Riehl and Verity [33, p. xii] specifically highlight this construction as remarkably subtle in ∞-category theory. It is for this reason that we introduced $\langle \mathsf{tw} \mid -\rangle$. Recall the visualization of $\langle \flat \mid \Delta^n \to \langle \mathsf{tw} \mid C\rangle\rangle$:

$$\begin{array}{ccccccc} c_n & \leftarrow & c_{n-1} & \leftarrow & \cdots & \leftarrow & c_0 \\ \downarrow & & & & & & \\ c_{n+1} & \rightarrow & c_{n+2} & \rightarrow & \cdots & \rightarrow & c_{2n} \end{array} \tag{1}$$

The projection to $\langle \mathsf{op} \mid C\rangle$ gives the top row and the map to C yields the bottom. This visualization for n-simplices is very similar to that of $C^{\mathbb{I}} = \sum_{c_0,c_1} \hom(c_0, c_1)$, but the top row has been twisted to ensure that one restriction lands in $\langle \mathsf{op} \mid C\rangle$ as required for a bifunctorial version of $\hom(-,-)$:

Theorem 3.4.* *The function $\langle \mathsf{tw} \mid C\rangle \to \langle \mathsf{op} \mid C\rangle \times C$ given by $\langle \pi_0^{\mathsf{tw}}, \pi_1^{\mathsf{tw}}\rangle$ is covariant, and the induced map $\langle \mathsf{op} \mid C\rangle \times C \to \mathcal{S}$ is the desired function Φ.*

The proof of covariance goes via Lemma 2.28 and Axiom G to verify that an n-simplex of $\langle \mathsf{tw} \mid C\rangle$ as in (1) indeed consists of an n-simplex of $\langle \mathsf{op} \mid C\rangle$ (the top row), an n-simplex of C (the bottom row), and a 0-simplex $\langle \mathsf{tw} \mid C\rangle$ (the left arrow).

Notation 3.5. We write $\Phi_D : \langle \mathsf{op} \mid D\rangle \times D \to \mathcal{S}$ for the same construction applied to some category D. Within this section, we continue to write Φ as shorthand for Φ_C.

Corollary 3.6. *If $c_0 : \langle \mathsf{op} \mid C\rangle$ and $c_1 : C$, then $\Phi(c_0, c_1) = \Phi_{\langle \mathsf{op} \mid C\rangle}(\mathsf{mod}_{\mathsf{op}}(\mathsf{mod}_{\mathsf{op}}(c_1)), c_0)$.*

3.2 The Yoneda lemma

With a bi-functorial version of $\hom(-,-)$ to hand, we can now straightforwardly define the Yoneda embedding $\mathbf{y}$ and leverage Lemma 3.3 into a result about $\mathbf{y}$:

Definition 3.7 (Yoneda). $\mathbf{y} = \lambda c.\, \Phi(-, c) : C \to \widehat{C}$.

Lemma 3.8. $\hom(\mathbf{y}(c), X) \cong X(\mathsf{mod}_{\mathsf{op}}(c))$ *for all $X : \widehat{C}$ and $c :_\flat C$.*

Proof. Since c is $\flat$-annotated, using Theorem 3.4 and Corollary 3.6 we have the following identification $\hom_{\langle \mathsf{op} \mid C\rangle}(\mathsf{mod}_{\mathsf{op}}(c), -) = \Phi(-, c)$. Moreover, by Lemma 3.1 we additionally have the following:

$$\prod_{c' : \langle \mathsf{op} \mid C\rangle} \Phi(c', c) \to X(c') \cong \hom(\mathbf{y}(c), X)$$

The conclusion now follows by Lemma 3.3. □

A great deal of category theory is contained within Lemma 3.8. It shows that $\mathbf{y}$ is fully-faithful on $\flat$-annotated elements of C and that C is a full subcategory of $\widehat{C}$:

Lemma 3.9. $\mathbf{y} : C \to \widehat{C}$ *induces an equivalence $C \simeq \widehat{C}_{\mathsf{isRepr}}$ where* $\mathsf{isRepr} = \lambda X.\, \sum_{c:C} X = \mathbf{y}(c)$.[6]

While Lemma 3.8 follows directly from Lemma 3.3, the above consequence can only be expressed once there exists a *category* of presheaves—something missing from Riehl and Shulman [1]. This opens up a new proof strategy: to prove a result of C, we first prove that it holds for $\mathcal{S}$, then $\widehat{C}$, then that it restricts to the full subcategory. For instance, we may prove the aforementioned characterization of natural isomorphisms:

[6] Note that $\mathsf{isRepr}(X)$ is a proposition due to Lemma 3.8 and Corollary 2.36.

Theorem 3.10. *If $C, D :_\flat \mathcal{U}$ are categories, $F, G :_\flat C \to D$, and $\alpha :_\flat \hom(F, G)$, then $\prod_{c:C} \mathsf{isIso}(\alpha\, c)$ if $\prod_{c:_\flat C} \mathsf{isIso}(\alpha\, c)$.*

Proof. By Lemma 3.9, it suffices to prove this for $D = \widehat{D_0}$. To show this, note that this theorem is trivial for $C = \Delta^0$ and for $C = \Delta^1, D = \mathcal{S}$ it is a consequence of Corollary 2.38. The Segal condition for $\mathcal{S}$ then implies the theorem for $C = \Delta^n, D = \mathcal{S}$.

To prove this theorem for C arbitrary and $D = \widehat{D_0}$, by Axiom D and Proposition 2.23 it suffices to show that $\left(\sum_{c:C} \mathsf{isIso}(\alpha\, c)\right) \to \left(\sum_{c:C} \langle \sharp \mid \mathsf{isIso}(\alpha\, c)\rangle\right)$ is an equivalence. By Axiom F, it suffices to prove for all n:

$$\mathsf{isEquiv}\Big(\big\langle \flat \mid \big(\textstyle\sum_{c:C} \mathsf{isIso}(\alpha\, c)\big)^{\Delta^n}\big\rangle \to \big\langle \flat \mid \big(\textstyle\sum_{c:C} \langle \sharp \mid \mathsf{isIso}(\alpha\, c)\rangle\big)^{\Delta^n}\big\rangle\Big)$$

Unfolding and commuting $\flat$ with $\sum$, it suffices to show that for every $c :_\flat \Delta^n \to C$ the following holds:

$$\textstyle\prod_{\sigma:\Delta^n} \mathsf{isIso}(\alpha(c\,\sigma)) \simeq \prod_{\sigma:_\flat \Delta^n} \mathsf{isIso}(\alpha(c\,\sigma))$$

Replacing α with $\alpha \circ c$, however, reduces us to the already proven case of $C = \Delta^n, D = \mathcal{S}$. □

Corollary 3.11. *The Yoneda embedding preserves all limits.*

Proof. If $F :_\flat I \to C$ and $\varprojlim F$ exists, then functoriality of $\mathbf{y}$ induces a map $\mathbf{y}(\varprojlim F) \to \varprojlim(\mathbf{y} \circ F)$, so it suffices to check that this map is invertible at all $c :_\flat C$. Unfolding, we must argue that $\hom(c, \varprojlim F) \simeq \varprojlim \hom(c, F)$ is an equivalence, but this is immediate by definition. □

Clearly Lemma 3.8 is already powerful. However, it does not capture that this equivalence is *natural* in both c and X—or, more precisely, since c is $\flat$-annotated and the equivalence is in $\mathcal{U}$, the naturality it yields is trivial. We are able to prove a far stronger version of the Yoneda lemma that (1) does not need to assume that $c :_\flat C$, and (2) yields the desired functoriality in both c and X. To do so, we replace $\hom(-,-)$ with Φ:

Theorem 3.12.* (Functorial Yoneda lemma) *There is a natural isomorphism $\Phi_{\widehat{C}}(\mathbf{y}^\dagger(-), -) \cong \mathsf{eval} : \langle \mathsf{op} \mid C\rangle \times \widehat{C} \to \mathcal{S}$.*

4 Revisiting adjunctions

With presheaves and the Yoneda embedding available, we now revisit the theory of adjoint functors introduced by Riehl and Shulman [1] in STT. They define a pair of functions $f : C \to D$ and $g : D \to C$ to be adjoint when equipped with $\iota : \prod_{c,d} \hom(f(c), d) \simeq \hom(c, g(d))$. While they produce several equivalent reformulations using a unit and counit natural transformations, no non-trivial examples of adjunctions are given—unsurprisingly, since concrete examples of categories in STT are relatively recent. Even with $\mathcal{S}$ available it is quite difficult to produce examples of such adjunctions.

It is far more feasible to construct only f and then show that $\Phi(f^\dagger(-), d) : \widehat{C}$ is representable for every $d :_\flat D$. This is comparable to Theorem 3.10: we wish to give a functorial definition of either f or g and a *non-functorial* definition of the other, and then show that this can be upgraded to a full adjunction. In this section, we show that this is indeed possible, and we observe that a number of important adjunctions and results are then immediately within reach. In particular, we shall use this technique to prove that $\widehat{C}$ is cocomplete and, moreover, is the free cocompletion of C.

4.1 Pointwise adjunctions to adjunctions

Let us begin by formalizing the notion of pointwise adjoints:

Definition 4.1. We say that $f :_\flat C \to D$ is a pointwise left adjoint if $\prod_{d :_\flat D} \mathsf{isRepr}(\Phi(f^\dagger(-), d))$; f is a pointwise right adjoint if $f^\dagger : \langle \mathsf{op} \mid C \rangle \to \langle \mathsf{op} \mid D \rangle$ is a pointwise left adjoint.

Our main theorem relies on two crucial preliminary results. The first shows that any pointwise left adjoint f gives rise to a function in the other direction picking out the various (necessarily unique) representing objects for $\Phi(f^\dagger(-), d)$.

Lemma 4.2. *If $f :_\flat C \to D$ is a pointwise left adjoint, then the type of morphisms $g :_\flat D \to C$ equipped with a natural isomorphism $\iota : \Phi(f^\dagger(-), -) \cong \mathbf{y} \circ g$ is contractible.*

Proof. Since $\mathbf{y}$ is an embedding, this type is a proposition. It therefore suffices to show that it is inhabited. By assumption, $\bar{g} = \Phi(f^\dagger(-), d)$ is representable for all $d :_\flat D$, and thus it factors through $\widehat{C}_{\mathsf{isRepr}}$. Post-composing with the equivalence $\widehat{C}_{\mathsf{isRepr}} \simeq C$ yields the desired $g : D \to C$. □

Using this, we prove a universal case of the theorem improving a pointwise adjoint to an adjoint: every $g :_\flat D \to C$ that is a cartesian fibration [14] such that the fiber over every $c :_\flat C$ has an initial object [12] admits a left adjoint.

Lemma 4.3. *If $g :_\flat D \to C$ is cartesian and for each $c :_\flat C$ the fiber D_c has an initial object, then there exists $f : C \to D$ such that $f(c)$ is initial in D_c for all $c : C$.*

Proof. Note that $\mathsf{hasInitialObj}(D_c)$ is a proposition and, therefore, by Axiom D we may assume $\langle \flat \mid \mathsf{hasInitialObj}(D_c) \rangle$ holds for each $c :_\flat C$. With this observation to hand, we can show that g is a pointwise right adjoint: if $c :_\flat C$, $d : D$:

$$\begin{aligned} &\Phi_C(\mathsf{mod}_{\mathsf{op}}(c), g(d)) \\ &\quad \simeq \hom(c, g(d)) \\ &\quad \simeq \hom(\mathbf{0}_{D_c}, d) && g \text{ is cartesian} \\ &\quad \simeq \Phi_D(\mathsf{mod}_{\mathsf{op}}(\mathbf{0}_{D_c}), d) \end{aligned}$$

The last step uses our observation that $\langle \flat \mid \mathsf{hasInitialObj}(D_c) \rangle$ and not only $\mathsf{hasInitialObj}(D_c)$ holds.

Accordingly, we obtain a function $f :_\flat C \to D$ which sends $c :_\flat C$ to $\mathbf{0}_{D_c}$. It remains to show that $f(c)$ is initial in D_c for all $c : C$. Since $D = \sum_{c:C} D_c$, this amounts to the following map being an equivalence: $\left(\sum_{d:D} \hom_{D_{g(d)}}(f(g(d)), d)\right) \to D$.

To prove this, we use Theorem 2.34 which allows us to reduce to the $\flat$-annotated case, where the conclusion follows from the fact that $f(c)$ is then initial in D_c. □

Theorem 4.4. *Pointwise right adjoints are right adjoints.*

Proof. Given a map $g :_\flat D \to C$, consider the cartesian family

$$\pi : (C \downarrow g) = \left(\sum_{c:C} \sum_{d:D} \hom(c, g(d))\right) \to C$$

Since g is a pointwise right adjoint, each fiber of π over $c :_\flat C$ has an initial object. We then apply Lemma 4.3 to obtain $\bar{f} : C \to (C \downarrow g)$. Finally, the composite $\pi_2 \circ \bar{f}$ is the desired left adjoint to g:

$$\begin{aligned} &\hom_C(c, g(d)) \\ &\quad \simeq \sum_{\alpha:\hom_C(c,g(d))} \hom_{(C\downarrow g)_c}(\bar{f}(c), (c, d, \alpha)) \\ &\quad \simeq \sum_{\alpha:\hom_C(c,g(d))} \sum_{\beta:\hom_D(f(c),d)} g(\beta) \circ \pi_3(\bar{f}(c)) = \alpha \end{aligned}$$

$$\simeq \hom_D(f(c), d)$$

The first step uses the initiality of $\bar{f}(c)$ in the fiber over c and the second unfolds the definition of a morphism in $(C \downarrow g)$. □

4.2 Examples of adjunctions

We take advantage of Theorem 4.4 to produce vital examples of adjoints.[7] The most important is the following:

Theorem 4.5. *If $f :_\flat D \to C$ and D is small, then $\widehat{f} \coloneqq (f^\dagger)^* : \widehat{C} \to \widehat{D}$ is a right adjoint with left adjoint $f_!$.*

Proof. For notational simplicity, we replace C and D with $\langle \mathrm{op} \mid C \rangle$ and $\langle \mathrm{op} \mid D \rangle$. By Theorem 4.4, it suffices to assume $X :_\flat D \to \mathcal{S}$ and to construct $f_!(X) : C \to \mathcal{S}$ along with a natural bijection $\prod_Y \hom(f_!(X), Y) \simeq \hom(X, f^*(Y))$. To this end, we take $f_!(X)$ to be the following covariant family:

$$f_!(X)\, c = \bigcirc_{\mathrm{grpd}}\left(\textstyle\sum_{d:D} \hom(f(d), c) \times X(d)\right)$$

This family is covariant by a slight variation of an argument of Buchholtz and Weinberger [14, Proposition 5.2.20]. Finally, we note the following chain of equivalences:

$$\begin{aligned}
&\hom(f_!(X), Y) \\
&\simeq \textstyle\prod_{c:C} f_!(X)(c) \to Y(c) \\
&\simeq \textstyle\prod_{c:C} \bigcirc_{\mathrm{grpd}}\left(\sum_{d:D} \hom(f(d), c) \times X(d)\right) \to Y(c) \\
&\simeq \textstyle\prod_{c:C}\left(\sum_{d:D} \hom(f(d), c) \times X(d)\right) \to Y(c) \\
&\simeq \textstyle\prod_{d:D} X(d) \to \prod_{c:C} \hom(f(d), c) \to Y(c) \\
&\simeq \textstyle\prod_{d:D} X(d) \to Y(f(d)) \\
&\simeq \hom(X, f^*(Y))
\end{aligned}$$

□

Corollary 4.6. *The left adjoint $f_! \dashv \widehat{f}$ satisfies $f_! \circ \mathbf{y} \cong \mathbf{y} \circ f$.*

Corollary 4.7. *$\mathcal{S}$ is small cocomplete:* const $: \mathcal{S} \to \mathcal{S}^C$ *is a right adjoint with left adjoint* $\varinjlim$ *for small categories $C :_\flat \mathcal{U}$. Explicitly, if $X :_\flat C \to \mathcal{S}$, then $\varinjlim_C X = \bigcirc_{\mathrm{grpd}}\left(\sum_{c:C} X(c)\right)$.*

Remark 4.8. One can prove $\mathcal{S}$ is complete (that const $\dashv \varprojlim$) and more generally that $\widehat{f}$ is a right adjoint by a result of Gratzer et al. [2]. In particular, they show $\prod_{c:C} X(c) : \mathcal{S}$ whenever $X : C \to \mathcal{S}$ and $C :_\flat \mathcal{U}$. Corollary 2.38 and Lemma 3.1 then imply that $\hom\left(A, \prod_{c:C} X(c)\right) \simeq \hom(\mathrm{const}\, A, X)$.

The following lemma does not require Theorem 4.4, but is merely a consequence of manipulating natural transformations:

Lemma 4.9. *If $f : C \to D$ is an adjoint so is $f_* : C^A \to D^A$.*

Corollary 4.10. *If C is (co)complete so is C^D and (co)limits are computed pointwise. In particular, $\widehat{C}$ is (co)complete.*

Finally, we show the full subcategories $\mathcal{S}_{\leq n}$ of $\mathcal{S}$ defined by n-truncated types form reflective subcategories of $\mathcal{S}$. The idea is simple: use the truncation HITs. However, it is not automatic that they restrict to $\|-\|_n : \mathcal{S} \to \mathcal{S}_{\leq n}$. We prove this alongside with the reflectivity of $\mathcal{S}_{\leq n}$ using Theorem 4.4:

[7] As the slogan goes: "adjoints arise everywhere"

Corollary 4.11. *The inclusion* $\mathcal{S}_{\leq n} \to \mathcal{S}$ *is a right adjoint.*

Corollary 4.12. $\mathcal{S}_{\leq n}$ *is (co)complete.*

The same methodology applies to the subcategory of modal types associated to an idempotent monad [8].

4.3 The universal property of presheaf categories

Next, we generalize Theorem 4.5 to show that if $f :_\flat C \to E$ where C is a small category and E is a cocomplete category, then $\Phi(f^\dagger(-), -) : E \to \widehat{C}$ is a right adjoint loosely following the argument given by Cisinski [20]. We begin with a few general lemmas. In what follows, fix C and E as above.

First, as a corollary of the proof of Theorem 4.5:

Lemma 4.13. *The colimit of* $\mathbf{y} : C \to \widehat{C}$ *is* $\mathbf{1}_{\widehat{C}} = \lambda\_.\,\mathbf{1}$.

From the above, and further inspection of colimits, we are able to derive a result of independent interest: Every presheaf is the colimit of representable presheaves.

Lemma 4.14 (Density of $\mathbf{y}$). *If* $X :_\flat \widehat{C}$, *then* $X \cong \varinjlim_{\langle \mathsf{op} | \widetilde{X} \rangle} \mathbf{y} \circ \pi^\dagger$, *where* $\widetilde{X} = \sum_{c : \langle \mathsf{op} | C \rangle} X(c)$.

Proof. We begin with the following computation where $\pi : \widetilde{X} \to \langle \mathsf{op} \mid C \rangle$ and $\pi_!^\dagger : \mathcal{S}^{\widetilde{X}} \to \widehat{C}$:

$$\pi_!^\dagger \mathbf{1} \cong \pi_!^\dagger \big(\varinjlim_{\langle \mathsf{op} | \widetilde{X} \rangle} \mathbf{y}\big) \cong \varinjlim_{\langle \mathsf{op} | \widetilde{X} \rangle} \pi_!^\dagger \circ \mathbf{y} \cong \varinjlim_{\langle \mathsf{op} | \widetilde{X} \rangle} \mathbf{y} \circ \pi^\dagger$$

We have used the fact that $\pi_!^\dagger$, a left adjoint, commutes with colimits [12]. To show $\pi_!^\dagger \mathbf{1} \cong X$, we note that for all $Z : \widehat{C}$:

$$\begin{aligned} \hom(\pi_!^\dagger \mathbf{1}, Z) &\simeq \hom_{\widetilde{X} \to \mathcal{S}}(\mathbf{1}, Z \circ \pi) \\ &\simeq \textstyle\prod_{(c,x) : \sum_{c : \langle \mathsf{op} | C \rangle} X(c)} Z(c) \\ &\simeq \textstyle\prod_{c : \langle \mathsf{op} | C \rangle} X(c) \to Z(c) \\ &\simeq \hom(X, Z) \end{aligned}$$

The conclusion now follows from the Yoneda lemma. □

Lemma 4.15. $\mathbf{n}_f = \Phi(f^\dagger(-), -) : E \to \widehat{C}$ *is a right adjoint.*

Proof. We will prove that $\mathbf{n}_f$ is a pointwise right adjoint. Accordingly, fixing $X :_\flat \widehat{C}$ we must construct $e : \langle \mathsf{op} \mid E \rangle$ such that $\Phi(e, -) \cong \Phi(\mathsf{mod}_{\mathsf{op}}(X), \mathbf{n}_f(-))$. Since $X \cong \varinjlim_{\langle \mathsf{op} | \widetilde{X} \rangle} \mathbf{y} \circ \pi^\dagger$ and E is cocomplete, by the dual of Corollary 3.11 it suffices to assume $\mathsf{mod}_{\mathsf{op}}(X) = \mathbf{y}^\dagger(c)$ with $c : \langle \mathsf{op} \mid C \rangle$.[8] Finally, take $e = f^\dagger(c)$ and $\Phi(f^\dagger(c), -) \cong \Phi(\mathbf{y}^\dagger(c), \mathbf{n}_f(-))$ by Theorem 3.12. □

We are now able to prove, as promised, the universal property of $\widehat{C}$. If we write $\mathsf{CC}(\widehat{C}, E)$ for the full subcategory of $\widehat{C} \to E$ spanned by functors preserving all colimits, then $\mathbf{y}^* : \mathsf{CC}(\widehat{C}, E) \to (C \to E)$ is an equivalence. To prove this, we essentially argue that there is a map sending f to the left adjoint to $\mathbf{n}_f$ and that this is the inverse to $\mathbf{y}^*$.

Theorem 4.16.* $\mathbf{y}^* : \mathsf{CC}(\widehat{C}, E) \to (C \to E)$ *is an equivalence.*

[8] Note the lack of $\flat$-annotation here: we must ensure that we are functorial in c in order to obtain a *diagram* in E.

5 The theory of Kan extensions

A unifying concept in category theory are *Kan extensions*, which are universal extensions of functors along functors on the same domain. Mac Lane, one of the founders of category theory, famously stated: "The notion of Kan extensions subsumes all the other fundamental concepts of category theory," such as (co)limits and adjunctions [7, 34].

Definition 5.1 (Kan extensions). Given a map $f : C \to D$ and a category E, the left (right) Kan extension lan_f (ran_f) is the left (right) adjoint to $f^* : E^D \to E^C$.

While the definition makes sense in general, to use the results of the previous sections, we shall assume $f :_\flat C \to D$ and $E :_\flat \mathcal{U}$. In Section 5.1 we show that Kan extensions exist whenever E is (co)complete and in Sections 5.2 and 5.3 we put this to work by deducing two important results: Quillen's theorem A and the properness of cocartesian fibrations. Our arguments for the existence of Kan extensions and Quillen's theorem A adapt the (model-agnostic) ∞-categorical arguments of Ramzi [35].

5.1 Existence and characterization of Kan extensions

We can prove that Kan extensions can be computed in an expected way. For $d : D$, we write $C_{/d} :\equiv C \times_D D_{/d}$ and $C_{d/} :\equiv C \times_D D_{d/}$. We assume that C and D are both small so each $C_{/d}$ is also small. By Theorem 4.5 and Lemma 4.9:

Lemma 5.2. *If $E = \widehat{A}$ for some category $A :_\flat \mathcal{U}$, then lan_f exists. Moreover, if $X :_\flat C \to E$ and $d : D$, then $\mathsf{lan}_f\ X\, d = \varinjlim(C_{/d} \to C \to E) = \bigcirc_{\mathsf{grpd}}\big(\sum_{(c,\_):C_{/d}} X(c)\big)$.*

This yields more generally:

Theorem 5.3. *If E is cocomplete, then lan_f exists, and if $X :_\flat C \to E$, $d : D$, then $\mathsf{lan}_f\ X\, d = \varinjlim(C_{/d} \to C \to E)$.*

Proof. It suffices to argue that precomposition is pointwise a right adjoint and so we fix $X :_\flat C \to E$. By Theorem 4.16, we may view X as the composition $\bar{X} \circ \mathbf{y}$, where $\bar{X} : \widehat{C} \to E$ is the left adjoint to $\mathbf{n}_X$. Next, we observe by Lemma 5.2 that $\mathbf{y} : C \to \widehat{C}$ admits an extension to D along f, namely $\mathsf{lan}_f\, \mathbf{y} : D \to \widehat{C}$, and we claim that $\bar{X} \circ \mathsf{lan}_f\, \mathbf{y}$ is our desired extension of f. Fixing $Z : D \to E$, we calculate:

$$\begin{aligned} \hom_{D\to E}(\bar{X} \circ \mathsf{lan}_f\, \mathbf{y}, Z) &\simeq \hom_{D\to\widehat{C}}(\mathsf{lan}_f\, \mathbf{y}, \mathbf{n}_X \circ Z) \\ &\simeq \hom_{C\to\widehat{C}}(\mathbf{y}, \mathbf{n}_X \circ Z \circ f) \\ &\simeq \hom_{C\to E}(\bar{X} \circ \mathbf{y}, Z \circ f) \\ &= \hom_{C\to E}(X, Z \circ f) \end{aligned}$$

The expected colimit formula continues to hold as a consequence of Lemma 5.2 and the cocontinuity of $\bar{X}$. □

By duality, we obtain the following variant:

Theorem 5.4. *If E is complete, then ran_f exists and is specified by the dual limit formula: $\mathsf{ran}_f\ X\, d = \varprojlim(C_{d/} \to C \to E)$.*

5.2 Cofinal functors

It is frequently useful to reduce show that the limit of a complex diagram D can be calculated by first restricting to a simpler diagram using $f : C \to D$ and calculating the limit there e.g., restricting from $\mathbb{Z}$ to $\mathbb{Z}_{\leq 0}$. When this approach is valid, f is said to be (left) cofinal:[9]

Definition 5.5. A functor $f :_\flat C \to D$ is *left cofinal* if for every $X :_\flat D \to \mathcal{S}$ the map $\varprojlim_D X \to \varprojlim_C X \circ f$ is an equivalence. A map is *right cofinal* if its opposite is left cofinal.

While this definition is asymmetrical in its treatment of left and right, we shall restore the symmetry as a consequence of Quillen's Theorem A in the next section, see Corollary 5.15.

Recall that $\varprojlim_D X = \prod_{d:D} X(d)$ and so the definition of left cofinality equivalently states that the restriction map $\left(\prod_{d:D} X(d)\right) \to \left(\prod_{c:C} X(f(c))\right)$ is an equivalence.

Example 5.6. The $\{0\}/\{1\}$ inclusion $\mathbf{1} \to \mathbb{I}$ is left/right cofinal.

Lemma 5.7. *If $f :_\flat C \to D$ is left cofinal and $X :_\flat D \to E$, then $\varprojlim_C (X \circ f)$ and $\varprojlim_D X$ both exist whenever either exists and are canonically isomorphic.*

Proof. By Corollary 3.11, we replace E with $\widehat{E}$ and by Corollary 4.10 we reduce to $\mathcal{S}$ where the result is immediate. □

Lemma 5.8.* *If $C :_\flat \mathcal{U}$, then $\bigcirc_{\mathrm{grpd}} C \simeq \bigcirc_{\mathrm{grpd}} \langle \mathrm{op} \mid C \rangle$.*

Lemma 5.9. *For every C, the canonical map $C \to \bigcirc_{\mathrm{grpd}} C$ is both left and right cofinal.*

Proof. By Lemma 5.8, it suffices to argue that this map is left cofinal. To this end, we must show the following map to be an equivalence for every $X : \bigcirc_{\mathrm{grpd}} C \to \mathcal{S}$:

$$\left(\prod_{d:\bigcirc_{\mathrm{grpd}} C} X(d)\right) \to \left(\prod_{c:C} X(\eta(c))\right)$$

However, $X(d)$ is discrete for every $d : \bigcirc_{\mathrm{grpd}} C$ and so this is simply the universal property of $\bigcirc_{\mathrm{grpd}}$. □

Corollary 5.10. *If $\bigcirc_{\mathrm{grpd}} C = \mathbf{1}$, then $\varprojlim_C A = A$ for $A : \mathcal{S}$.*

5.3 Quillen's Theorem A

Our next goal is to prove the ∞-categorical version of Quillen's theorem A. Unlike traditional proofs, we follow Ramzi [35] and rely on having already established the basic apparatus of Kan extensions to simplify our argument.

Theorem 5.11. *A functor $f :_\flat C \to D$ is right cofinal if and only if $\bigcirc_{\mathrm{grpd}}(C_{d/}) \simeq \mathbf{1}$ for all $d :_\flat D$* (Quillen right cofinal).

Remark 5.12. This result shows that, in particular, cofinality doesn't depend on the particular universe $\mathcal{S}$ chosen.

Lemma 5.13. *If $f :_\flat C \to D$ is Quillen right cofinal and $X :_\flat D \to \widehat{A}$, then $\varinjlim_D X \simeq \varinjlim_C X \circ f$.*

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9 The terminology comes from the fact that cofinal (in the classical sense) inclusions of partial orders form right cofinal functors.

Proof. This statement is pointwise, so we quickly reduce to $\mathcal{S}$ instead of $\widehat{A}$. In this situation, we wish to show that the following commutes:

$$\begin{array}{ccc} \mathcal{S}^D & \xrightarrow{f^*} & \mathcal{S}^C \\ & {\scriptstyle \varinjlim_D} \searrow \quad \swarrow {\scriptstyle \varinjlim_C} & \\ & \mathcal{S} & \end{array}$$

Note that all three morphisms are left adjoints, and so it suffices to compare their right adjoints: the constant functors Δ_C and Δ_D, along with the right Kan extension ran_f. We next note that there is at least a comparison map $\Delta_D \to \mathsf{ran}_f \circ \Delta_C$ given by transposing the identity map $f^* \circ \Delta_D \to \Delta_C$. We must argue that this map is pointwise invertible, and so we reduce to considering $X :_\flat \mathcal{S}$ and $d :_\flat D$, and we must show the following, using Theorem 5.4: $X \simeq \varprojlim_{C_{d/}} X$. This now follows from our assumption and Corollary 5.10. □

Lemma 5.14. *If $f :_\flat C \to D$ is Quillen right cofinal, E is cocomplete, and $X :_\flat D \to E$, then $\varinjlim_D X \simeq \varinjlim_C X \circ f$.*

Proof. We reduce to the case where $E = \widehat{D}$ (and therefore Lemma 5.13) by factoring X as $\bar{X} \circ \mathbf{y}$ and noting that $\bar{X}$ preserves colimits by construction. □

Proof of Theorem 5.11. To see that Quillen right cofinality implies right cofinality, we apply Lemma 5.14 to $\langle \mathsf{op} \mid \mathcal{S} \rangle$, and calculate:

$$\varprojlim_{\langle \mathsf{op} | D \rangle} X \simeq \varinjlim_D X^\dagger \simeq \varinjlim_C X^\dagger \circ f \simeq \varprojlim_{\langle \mathsf{op} | C \rangle} X \circ f^\dagger$$

For the reverse, suppose that f is right cofinal. We note that by the dual of Lemma 5.7 (again applied to $\langle \mathsf{op} \mid \mathcal{S} \rangle$), the canonical map $\varinjlim_C X \circ f \to \varinjlim_D X$ is an equivalence for any $X :_\flat D \to \mathcal{S}$. Fix $d :_\flat D$ and choose $X = \hom(d, -) = \Phi(\mathsf{mod}_{\mathsf{op}}(d), -)$ such that the colimits in question are precisely $\bigcirc_{\mathsf{grpd}} D_{d/}$ and $\bigcirc_{\mathsf{grpd}} C_{d/}$, using Theorem 4.5. This completes the proof since $\bigcirc_{\mathsf{grpd}} D_{d/} = \mathbf{1}$. □

Corollary 5.15. *A functor $f :_\flat C \to D$ is right cofinal if and only if, for every $X :_\flat D \to \mathcal{S}$ the map $\varinjlim_D X \to \varinjlim_C X \circ f$ is an equivalence.*

This restores the symmetry between left and right cofinal functors, as promised. The following alternative characterization is also often useful:

Proposition 5.16.* *$f :_\flat C \to D$ is left (right) cofinal if and only if for every covariant (contravariant) family $\pi :_\flat X \to Y$, f is left orthogonal to π, i.e.,* $\mathsf{isEquiv}(X^D \to X^C \times_{Y^C} Y^D)$.

As another consequence we get the dual of Theorem 5.11:

Corollary 5.17. *A functor $f :_\flat C \to D$ is left cofinal if and only if $\bigcirc_{\mathsf{grpd}}(C_{/d}) \simeq \mathbf{1}$ for all $d :_\flat D$* (Quillen left cofinal).

We demonstrate the utility of Theorem 5.11 by giving a new and far simpler proof that cocartesian fibrations are *proper.*

Definition 5.18. A functor $\pi :_\flat E \to B$ between categories is *proper* if for all pullbacks (of $\flat$-functors) of the following form, v is right cofinal if u is right cofinal:

$$\begin{array}{ccccc} E'' & \xrightarrow{v} & E' & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow {\scriptstyle \pi'} & \lrcorner & \downarrow {\scriptstyle \pi} \\ B'' & \xrightarrow[u]{} & B' & \longrightarrow & B \end{array}$$

We call π *smooth* if $\pi^\dagger : \langle \mathrm{op} \mid E \rangle \to \langle \mathrm{op} \mid B \rangle$ is proper.

Lemma 5.19. *Smooth and proper functors are closed under composition and pullback.*[10]

Theorem 5.20. *Any (co)cartesian fibration is (proper) smooth.*

Proof. It suffices to treat the proper case. Fix a cocartesian fibration $\pi :_\flat E \to B$ and note that since cocartesian fibrations are stable under pullbacks, it suffices to that v is right cofinal in the following pullback diagram if u is right cofinal:

$$\begin{array}{ccc} A \times_B E & \xrightarrow{\;\;v\;\;} & E \\ \big\downarrow & \lrcorner & \big\downarrow \pi \\ A & \xrightarrow[\;\;u\;\;]{} & B \end{array}$$

We now use Theorem 5.11. For $e :_\flat E$ we compute the fiber:

$$\begin{aligned} &(A \times_B E) \times_E E_{e/} \\ &\quad \simeq A \times_B E_{e/} \\ &\quad \simeq A \times_B \left(\textstyle\sum_{b':B, f:\hom(\pi(e),b')} (E_{b'})^{\mathbb{I}} \right) && \pi \text{ is cocartesian} \\ &\quad \simeq \textstyle\sum_{(a,f):A \times_B B_{\pi(e)/}} (E_{u(a)})_{f_! e/} \end{aligned}$$

Applying $\bigcirc_{\mathrm{grpd}}$ to each fiber yields $\bigcirc_{\mathrm{grpd}} (E_{u(a)})_{f_! e/} \simeq \mathbf{1}$ (as coslices have initial elements) and $\bigcirc_{\mathrm{grpd}}(A \times_B B_{\pi(e)/}) \simeq \mathbf{1}$ since u is right cofinal by assumption. This implies that applying $\bigcirc_{\mathrm{grpd}}$ to the entire $\sum$-type produces $\mathbf{1}$ [8]. □

Corollary 5.21. *If $\pi :_\flat E \to B$ is cocartesian and $X :_\flat E \to D$, then the left Kan extension $\mathsf{lan}_\pi X$ sends $b :_\flat B$ to $\varinjlim(E_b \to E \to D)$.*

6 Conclusions and future work

We have introduced and studied the impact of the ∞-categorical Yoneda embedding in STT. This includes the development of classical concepts (Kan extensions, adjoints, (co)limits, etc.), all in the synthetic ∞-categorical setting. While some of the basic theory had been investigated in STT already, we were able to produce the first non-trivial concrete examples of, e.g., adjunctions (Theorem 4.5) and give several more refined versions of existing theorems (Theorem 3.10) which more closely match their standard counterparts.

6.1 Related work

There are several closely related type-theoretic approaches to synthetic (∞-)category theory. We may roughly divide these into (1) directed type theory, where every type is a category but various operations ($\prod$) must be restricted, and (2) variations on simplicial type theory. For instance, many directed type theories have been proposed and studied over the years [9, 10, 36–45]. In general, while these type theories are a promising approach to formalize category theory in type theory, none of them have thus far received as much attention as STT and, consequently, none have developed category theory to the extent of this work. Furthermore, it is substantially harder to design a directed type theory in this style (as it is a more radical alteration of the basic rules of type theory) and most proposals handle only 1-category theory rather than $(\infty, 1)$-categories. We note, however, that some of these type theories do include a version of Theorem 3.12 in the form of

[10]The definition of properness is formulated specifically to bake in the latter.

directed path induction [37–39]. Given, however, that few of our arguments rely on types which are not categories, we expect many of them to transfer to sufficiently rich future variants of directed type theory.

Other variations of simplicial type theory have been considered in the literature. For instance, several papers use additional judgmental structure (extension types) to get more definitional equalities around hom-types [1, 12–16] at the cost of making the interval a second-class type similar to two-level type theory [46, 47]. Other versions have favored a cubical interval [2] or even a cubical interval atop a cubical version of HoTT [9, 10]. Aside from the addition of modalities, our version of STT is deliberately minimalistic: we use only ordinary HoTT with a handful of postulates. Accordingly, our results can be interpret into essentially any incarnation of modal STT and does not rely on extra definitional equalities.

Finally, there are many attempts to formulate more conceptual and synthetic foundations for ∞-category theory which do not rely on type theory. For instance, the ∞-cosmos program of Riehl and Verity [33] aims to give a systematic account of the formal category theory and model-independence using 2-category theory. On the other hand, most practitioners in the field attempt to give looser "model independent" arguments which avoid relying on explicit computations as much as possible. We have successfully translated some of these arguments into our framework, proving that this informal discipline is effective (e.g., Section 5). More recently, Cisinski et al. [48] have begun to redevelop ∞-category theory in a deliberately informal and high-level language, splitting the difference between a formal theory like STT and the usual "model-independent" discipline of practitioners. We expect that their arguments can be translated into STT and we have shown that some of their primitive axioms are *provable* in STT (e.g., Axiom/Proposition 2.37, Proposition 2.35, and Lemma 4.3).

6.2 Future work

Many promising avenues for future work remain to be explored. While we have focused on presheaf categories and immediate consequences of their theory, we plan to port other foundational results from category theory (presentable and accessible categories, Bousfield localizations, topos theory, etc.) into STT. It would also be desirable to adapt more parts of the internal ∞-category theory and ∞-topos theory of Martini and Wolf [49–55] to STT. Additionally, we hope to extend a proof assistant like Agda [56] with the necessary support for modalities to give machine-checked versions of the proofs in this paper. On the foundational side, STT presently uses relies on a handful of axioms (Appendix B) and therefore satisfies only normalization and not canonicity. In future work, we hope to examine which of these principles can be given computational interpretations and to what extent one can 'compute' with synthetic ∞-categories.

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A The formal rules of MTT

The formal syntax of MTT is comprised of four judgments: $\vdash \Gamma$, $\Gamma \vdash \delta : \Delta$, $\Gamma \vdash a : A$, and $\Gamma \vdash A$. We list the relevant novel rules for these judgments below:

$$\boxed{\vdash \Gamma}$$

$$\frac{}{\vdash \mathbf{1}} \qquad \frac{\vdash \Gamma}{\vdash \Gamma.\{\mu\}} \qquad \frac{\vdash \Gamma \qquad \Gamma.\{\mu\} \vdash A}{\vdash \Gamma.(\mu \mid A)}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma.\{\mathsf{id}\} = \Gamma \qquad \vdash \Gamma.\{\mu\}.\{\nu\} = \Gamma.\{\mu \circ \nu\}}$$

$$\boxed{\Gamma \vdash \delta : \Delta}$$

$$\frac{}{\Gamma \vdash ! : \mathbf{1}} \qquad \frac{\vdash \Gamma \qquad \Gamma.\{\mu\} \vdash A}{\Gamma.(\mu \mid A) \vdash \uparrow : \Gamma} \qquad \frac{\Gamma \vdash \delta : \Delta \qquad \Gamma.\{\mu\} \vdash a : A}{\Gamma \vdash \delta.a : \Delta.(\mu \mid A)} \qquad \frac{\Gamma \vdash \delta : \Delta}{\Gamma.\{\mu\} \vdash \delta.\{\mu\} : \Delta.\{\mu\}}$$

$$\frac{\vdash \Gamma \qquad \mu \leq \nu}{\Gamma.\{\nu\} \vdash \Gamma.\{\mu \leq \nu\} : \Gamma.\{\mu\}}$$

$$\frac{\Gamma \vdash \gamma : \mathbf{1}}{\Gamma \vdash ! = \gamma : \mathbf{1}} \qquad \frac{\Gamma \vdash \delta : \Delta \qquad \Gamma.\{\mu\} \vdash a : A}{\Gamma \vdash \uparrow \circ (\delta.a) = \delta : \Delta} \qquad \frac{\Gamma \vdash \delta : \Delta.(\mu \mid A)}{\Gamma \vdash (\uparrow \circ \delta).\mathbf{v}[\delta] = \delta : \Delta.(\mu \mid A)}$$

$$\frac{\Gamma \vdash \delta : \Delta}{\Gamma \vdash \delta.\{\mathsf{id}\} = \delta : \Delta \qquad \Gamma.\{\nu \circ \mu\} \vdash \delta.\{\nu \circ \mu\} = \delta.\{\nu\}.\{\mu\} : \Delta.\{\nu \circ \mu\}}$$

$$\frac{\vdash \Gamma}{\Gamma.\{\mu\} \vdash \Gamma.\{\mu \leq \mu\} = \mathsf{id} : \Gamma.\{\mu\} \qquad \Gamma.\{\xi\} \vdash \Gamma.\{\mu \leq \nu\} \circ \Gamma.\{\nu \leq \xi\} = \Gamma.\{\mu \leq \xi\} : \Gamma.\{\mu\}}$$

$$\frac{\Gamma \vdash \delta : \Delta \qquad \mu \leq \nu}{\Gamma.\{\nu\} \vdash \Delta.\{\mu \leq \nu\} \circ \delta.\{\mu\} = \delta.\{\mu\} \circ \Gamma.\{\mu \leq \nu\} : \Delta.\{\mu\}}$$

$$\frac{\vdash \Gamma \qquad \mu_0 \leq \nu_0 \qquad \mu_1 \leq \nu_1}{\Gamma.\{\nu_1 \circ \nu_0\} \vdash {\scriptstyle \substack{\Gamma.\{\mu_1 \leq \nu_1\}.\{\mu_0\} \circ \Gamma.\{\mu_0 \leq \nu_0\} \\ = \Gamma.\{\mu_1 \circ \mu_0 \leq \nu_1 \circ \nu_0\}}} : \Gamma.\{\mu_1 \circ \mu_0\}}$$

$$\boxed{\Gamma \vdash A}$$

$$\frac{\Gamma.\{\mu\} \vdash A}{\Gamma \vdash \langle \mu \mid A \rangle} \qquad \frac{\Gamma \vdash \delta : \Delta \qquad \Delta.\{\mu\} \vdash A}{\Gamma \vdash \langle \mu \mid A \rangle[\delta] = \langle \mu \mid A[\delta.\{\mu\}] \rangle}$$

$$\boxed{\Gamma \vdash a : A}$$

$$\frac{\Gamma.\{\mu\} \vdash A}{\Gamma.(\mu \mid A).\{\mu\} \vdash \mathbf{v} : A[\uparrow.\{\mu\}]} \qquad \frac{\Gamma.\{\mu\} \vdash a : A}{\Gamma \vdash \mathsf{mod}_\mu(a) : \langle \mu \mid A \rangle}$$

$$\frac{\Gamma.(\nu \mid \langle \mu \mid A \rangle) \vdash B \qquad \Gamma.(\nu \circ \mu \mid A) \vdash b : B[\uparrow.\mathsf{mod}_\mu(\mathbf{v})] \qquad \Gamma.\{\nu\} \vdash a : \langle \mu \mid A \rangle}{\Gamma \vdash \mathsf{let}\ \mathsf{mod}_\mu(-) \leftarrow a\ \mathsf{in}\ b : B[\mathsf{id}.a]}$$

$$\frac{\Delta.(\nu \mid \langle \mu \mid A \rangle) \vdash B \qquad \Delta.(\nu \circ \mu \mid A) \vdash b : B[\uparrow.\mathsf{mod}_\mu(\mathbf{v})] \qquad \Delta.\{\nu\} \vdash a : \langle \mu \mid A \rangle \qquad \Gamma \vdash \delta : \Delta}{\Gamma \vdash \begin{smallmatrix} \mathsf{let}\ \mathsf{mod}_\mu(-) \leftarrow a[\delta.\{\nu\}]\ \mathsf{in}\ b[(\delta \circ \uparrow).\mathbf{v}] \\ = (\mathsf{let}\ \mathsf{mod}_\mu(-) \leftarrow a\ \mathsf{in}\ b)[\delta] \end{smallmatrix} : B[\delta.a]}$$

$$\frac{\Gamma.\{\mu\} \vdash a : A \qquad \Gamma \vdash \delta : \Delta}{\Gamma \vdash \mathsf{mod}_\mu(a)[\delta] = \mathsf{mod}_\mu(a[\delta.\{\mu\}]) : \langle \mu \mid A[\delta.\{\mu\}] \rangle}$$

$$\frac{\Gamma \vdash \delta : \Delta \qquad \Gamma.\{\mu\} \vdash a : A[\delta.\{\mu\}] \qquad \Delta.\{\mu\} \vdash A}{\Gamma.\{\mu\} \vdash \mathbf{v}[\delta.a.\{\mu\}] = a : A[\delta.\{\mu\}]}$$

$$\frac{\Gamma.(\nu \mid \langle \mu \mid A \rangle) \vdash B \qquad \Gamma.(\nu \circ \mu \mid A) \vdash b : B[\uparrow.\mathsf{mod}_\mu(\mathbf{v})] \qquad \Gamma.\{\nu\} \vdash a : \langle \mu \mid A \rangle}{\Gamma \vdash (\mathsf{let}\ \mathsf{mod}_\mu(-) \leftarrow \mathsf{mod}_\mu(a)\ \mathsf{in}\ b) = b[\mathsf{id}.a] : B[\mathsf{id}.\mathsf{mod}_\mu(a)]}$$

B The complete list of axioms

Axiom A. *There is a set* $\mathbb{I}$ *that forms a bounded distributive lattice* $(0, 1, \vee, \wedge)$ *such that* $\prod_{i,j:\mathbb{I}} i \leq j \vee j \leq i$ *holds.*

Axiom B. *The map* $\mathsf{mod}_\mu(a) = \mathsf{mod}_\mu(b) \to \langle \mu \mid a = b \rangle$ *sending* refl *to* $\mathsf{mod}_\mu(\mathsf{refl})$ *is an equivalence for all* $a, b :_\mu A$.

Axiom C. *There is an equivalence* $\neg : \langle \mathsf{op} \mid \mathbb{I} \rangle \to \mathbb{I}$ *which swaps* 0 *for* 1 *and* $\vee$ *for* $\wedge$.

Axiom D. *If* $A :_\flat \mathcal{U}$, *then* $\langle \flat \mid A \rangle \to A$ *is an equivalence* (discrete) *if and only if* $A \to A^{\mathbb{I}}$ *is an equivalence* ($\mathbb{I}$-null).

Axiom E. *The canonical map* $\mathsf{Bool} \to \mathbb{I}$ *is injective and induces an equivalence* $\mathsf{Bool} \simeq \langle \flat \mid \mathbb{I} \rangle$.

Axiom F. $f :_\flat A \to B$ *is an equivalence if and only if the following holds:*

$$\prod_{n:_\flat \mathsf{Nat}} \mathsf{isEquiv}((f_*)^\dagger : \langle \flat \mid \Delta^n \to A \rangle \to \langle \flat \mid \Delta^n \to B \rangle)$$

Axiom G. *For every category* $C :_\flat \mathcal{U}$ *we have:*

- *Maps* $\pi_0^{\mathsf{tw}} : \langle \mathsf{tw} \mid C \rangle \to \langle \mathsf{op} \mid C \rangle$, $\pi_1^{\mathsf{tw}} : \langle \mathsf{tw} \mid C \rangle \to C$.
- *Equivalences* $\iota : \langle \flat \mid C^{\langle \mathsf{op} \mid \Delta^n \rangle \diamond \Delta^n} \rangle \simeq \langle \flat \mid \langle \mathsf{tw} \mid C \rangle^{\Delta^n} \rangle$ *and* $\tau : \langle \mathsf{tw} \mid C \rangle \simeq \langle \mathsf{tw} \mid \langle \mathsf{op} \mid C \rangle \rangle$.

We require that π_0^{tw}, π_1^{tw}, ι, *and* τ *be natural*[11] *and that the diagrams in Fig. 2 commute.*

The following *duality axiom* was first studied by Blechschmidt [57] and implies that, e.g., $\mathbb{I}$ is a category. We did not introduce it in the main body of the paper as it was not explicitly invoked in any of our proofs.

Axiom H. *If* A *is a finitely presented* $\mathbb{I}$*-algebra* (*i.e.,* A *is a bounded distributive lattice equivalent to* $\mathbb{I}[x_1, \ldots, x_n]$ *quotiented by finitely many relations*) *and* $\hom_{\mathbb{I}\mathsf{Alg}}(A, \mathbb{I})$ *is the type of* $\mathbb{I}$*-algebra homomorphisms, then the map* $\lambda a\, f.\, f(a) : A \to (\hom_{\mathbb{I}\mathsf{Alg}}(A, \mathbb{I}) \to \mathbb{I})$ *is an equivalence.*

C Selected details from omitted proofs

Theorem 2.34.* *If* $C, D :_\flat \mathcal{U}$ *are categories, then* $F :_\flat C \to D$ *is an equivalence if (1) the induced map* $\langle \flat \mid C \rangle \to \langle \flat \mid D \rangle$ *is surjective, and (2) for any* $c, c' :_\flat C$ *the map* $\hom(c, c') \to \hom(F(c), F(c'))$ *is an equivalence.*

[11] By this, we mean that there is a choice of path filling for each naturality square, but we do not insist that these paths be coherent.

Proof. Suppose (1) and (2) holds. We prove that F is an equivalence using Axiom F and fix $n :_\flat \mathsf{Nat}$ such that it suffices to show $\mathsf{isEquiv}(F^\dagger_* : \langle \flat \mid \Delta^n \to C\rangle \to \langle \flat \mid \Delta^n \to D\rangle)$.

If $n = 0$, then by (1) $F^\dagger_*$ is surjective and by (2) combined with the Rezk condition, it is an embedding. Accordingly, $F^\dagger_*$ is an equivalence in this case. The case for $n = 1$ is an immediate consequence of the cases for $n = 0$ along with (2). In general, since $C^{\Delta^n} \simeq C^{\Delta^1} \times_C \cdots \times_C C^{\Delta^1}$ by the Segal condition and likewise for D, and $\langle \flat \mid - \rangle$ commutes with pullbacks, the case for $n \geq 2$ follows from $n = 0, 1$. □

Theorem 3.4.* *The function* $\langle \mathsf{tw} \mid C\rangle \to \langle \mathsf{op} \mid C\rangle \times C$ *given by* $\langle \pi_0^{\mathsf{tw}}, \pi_1^{\mathsf{tw}}\rangle$ *is covariant, and the induced map* $\langle \mathsf{op} \mid C\rangle \times C \to \mathcal{S}$ *is the desired function* Φ*.*

Proof. Both the covariance of $\langle \mathsf{tw} \mid C\rangle \to \langle \mathsf{op} \mid C\rangle \times C$ and, for each $c :_\flat C$, the equality $\mathrm{hom}(c, -) = \Phi(\mathsf{mod}_{\mathsf{op}}(c), -)$ are consequences of Axiom G. For instance, to prove covariance we use Lemma 2.28 to reduce to showing the following equivalence:

$$\begin{aligned}&\langle \flat \mid \langle \mathsf{tw} \mid C\rangle^{\Delta^n}\rangle \to \\ &\quad \langle \flat \mid \langle \mathsf{tw} \mid C\rangle\rangle \times_{\langle \flat \mid \langle \mathsf{op} \mid C\rangle\rangle \times \langle \flat \mid C\rangle} \langle \flat \mid \langle \mathsf{op} \mid C\rangle^{\Delta^n}\rangle \times \langle \flat \mid C^{\Delta^n}\rangle\end{aligned}$$

We can unfold and restructure this using the fact that $\langle \flat \mid - \rangle$ commutes with pullbacks along with Axiom G to showing the following restriction map is an equivalence:

$$\begin{aligned}&\langle \flat \mid \Delta^{2n+1} \to C\rangle \\ &\quad \to \left\langle \flat \mid \Delta^n \amalg_1 \Delta^1 \amalg_1 \Delta^n \to C \right\rangle\end{aligned}$$

This, in turn, follows from the fact that the inclusion map $\Delta^n \amalg_1 \Delta^1 \amalg_1 \Delta^n \to \Delta^{2n+1}$ is orthogonal to the category C.

Now fix $c :_\flat C$. We first use covariance to construct a transformation $\alpha : \prod_{d:C} \mathrm{hom}_C(c, d) \to \Phi(\mathsf{mod}_{\mathsf{op}}(c), d)$ using the element $\alpha(c, \mathrm{id}_c) : \Phi(\mathsf{mod}_{\mathsf{op}}(c), c)$ corresponding to the morphism $\mathsf{mod}_\flat(\mathrm{id}_c) : \langle \flat \mid \Delta^1 \to C\rangle$. It then suffices to check that α gives an equivalence on total types, which again can be checked using Axiom F to show the following:

$$\left\langle \flat \mid \Delta^n \to \textstyle\sum_d \mathrm{hom}_C(c, d)\right\rangle \simeq \left\langle \flat \mid \Delta^n \to \textstyle\sum_d \Phi(\mathsf{mod}_{\mathsf{op}}(c), d)\right\rangle$$

This follows from Axiom G and calculation. □

Theorem 3.12.* (Functorial Yoneda lemma) *There is a natural isomorphism* $\Phi_{\widehat{C}}(\mathbf{y}^\dagger(-), -) \cong \mathsf{eval} : \langle \mathsf{op} \mid C\rangle \times \widehat{C} \to \mathcal{S}$*.*

Proof. The central difficulty in this proof is to find a map $\Phi_{\widehat{C}}(\mathbf{y}^\dagger(-), -) \to \mathsf{eval}$ which can then be checked to be an equivalence. To construct this map, we use the presentation of $\langle \mathsf{op} \mid C\rangle \times \widehat{C} \to \mathcal{S}$ as covariant families over $\langle \mathsf{op} \mid C\rangle \times \widehat{C}$. In particular, we consider the following pullback diagrams:

$$\begin{array}{ccccc}\tilde{\Phi}_C & \xrightarrow{\;v\;} & V & \longrightarrow & \langle \mathsf{tw} \mid \widehat{C}\rangle \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \langle \mathsf{op} \mid C\rangle \times C & \xrightarrow[\mathrm{id} \times \mathbf{y}]{} & \langle \mathsf{op} \mid C\rangle \times \widehat{C} & \xrightarrow[\mathbf{y}^\dagger \times \mathrm{id}]{} & \langle \mathsf{op} \mid \widehat{C}\rangle \times \widehat{C}\end{array}$$

$$\begin{array}{ccccc}
\langle \mathsf{tw} \mid C \rangle & \xrightarrow{w} & W & \longrightarrow & \sum_{A:\mathcal{S}} A \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\langle \mathsf{op} \mid C \rangle \times C & \xrightarrow[\mathrm{id} \times \mathbf{y}]{} & \langle \mathsf{op} \mid C \rangle \times \widehat{C} & \xrightarrow[\mathsf{eval}]{} & \mathcal{S}
\end{array}$$

The claim is then that $V \simeq W$. To show this, we argue that if we replace the composite $\langle \mathsf{tw} \mid C \rangle \to \langle \mathsf{op} \mid C \rangle \times C \to \langle \mathsf{op} \mid C \rangle \times \widehat{C}$ with free covariant family, then the maps $\bar{v}, \bar{w}$ induced by v and w are both equivalences. The conclusion then follows $\bar{v} \circ \bar{w}^{-1}$ is the desired equivalence.

We recall that a slight variation on the argument by Buchholtz and Weinberger [14, Proposition 5.2.20] shows that the relevant free covariant fibration $Z : \langle \mathsf{op} \mid C \rangle \times \widehat{C} \to \mathcal{U}$ is given as follows

$$\begin{aligned} &Z(c, X) = \\ &\quad \bigcirc_{\mathsf{grpd}} \left(\textstyle\sum_{c_0, c_1} \hom(c_0, c) \times \hom(\mathbf{y}(c_1), X) \times \Phi(c_0, c_1)\right) \end{aligned}$$

To show that e.g., v induces an equivalence, we must show that the following map is an equivalence:

$$Z(c, X) \to \hom(\mathbf{y}^{\dagger}(c), X)$$

We may use Theorem 3.10 and assume that there exists $c' :_{\flat} C$ such that $c = \mathsf{mod}_{\mathsf{op}}(c)$ and that $X :_{\flat} \widehat{C}$. Moreover, since the right-hand side is a groupoid, this map is uniquely induced by extending the canonical map of the following type:

$$\begin{aligned} &\left(\textstyle\sum_{c_0, c_1} \hom(c_0, c) \times \hom(\mathbf{y}(c_1), X) \times \Phi(c_0, c_1)\right) \\ &\quad \to \Phi(\mathbf{y}^{\dagger}(c), X) \simeq X(c') \end{aligned}$$

This map sends (c_0, c_1, f, α, t) to $\alpha\, c\, (\Phi(f, \mathsf{id})\, t)$ and one may check directly that the assignment $x \mapsto \eta(c, c', \mathsf{id}, \mathsf{id}, F_x)$ is a quasi-inverse to this map where $F_x : \hom(\mathbf{y}(c'), X)$ corresponds to $x : X(\mathsf{mod}_{\mathsf{op}}(c'))$ under Lemma 3.8. The case for w is similar. ☐

Theorem 4.16.* $\mathbf{y}^* : \mathsf{CC}(\widehat{C}, E) \to (C \to E)$ *is an equivalence.*

Proof. We use Theorem 2.34. If $f :_{\flat} C \to E$, then $f_! : \widehat{C} \to E$ satisfies $f_! \circ \mathbf{y} = f$ and so $\mathbf{y}^*$ essentially surjective:

$$\Phi((f_! \circ \mathbf{y})^{\dagger}(-), -) \cong \Phi(\mathbf{y}^{\dagger}(-), \mathbf{n}_f(-)) \cong \mathbf{n}_f = \Phi(f^{\dagger}(-), -)$$

Moreover, if $F :_{\flat} \mathsf{CC}(\widehat{C}, E)$, then $(F \circ \mathbf{y})_! \cong F$, so that $\mathbf{y}^*$ is a bijection on $\flat$-elements. Let us write $f = F \circ \mathbf{y}$. We first construct a comparison map $\hom(f_!, F)$ by constructing construct a natural transformation $\hom(\mathsf{id}, \mathbf{n}_f(F(-)))$. Currying, this is equivalent to constructing a natural transformation between maps $\langle \mathsf{op} \mid C \rangle \times \widehat{C} \to \mathcal{S}$ and, in this form, id is given by evaluation ϵ and $\mathbf{n}_F(F(-))$ is $\Phi(f(-), F(-))$. We can replace ϵ with $\Phi(\mathbf{y}(-), -)$ by Theorem 3.12 and $\Phi(f(-), F(-)) = \Phi(F(\mathbf{y}(-)), F(-))$ by definition. Accordingly, the relevant map is supplied by Φ_F. It is routine to check that this is pointwise an equivalence by Lemma 4.14.

Finally, we now show that $\mathbf{y}^*$ is fully faithful. To show that it is fully faithful, we must show that if $f, g :_{\flat} C \to E$, then $\hom(f_!, g_!) \cong \hom(f, g)$. Both sides are groupoids, so it suffices to consider $\flat$-annotated elements. If $\alpha :_{\flat} \hom(f, g)$, then by transposing we may regard α as an element of $\langle \flat \mid C \to E^{\mathbb{I}} \rangle$ and the previous observation ensures that this type is equivalent to $\langle \flat \mid \mathsf{CC}(\widehat{C}, E^{\mathbb{I}}) \rangle$ which yields the desired conclusion after transposing. ☐

Lemma 5.8.* *If* $C :_{\flat} \mathcal{U}$, *then* $\bigcirc_{\mathsf{grpd}} C \simeq \bigcirc_{\mathsf{grpd}} \langle \mathsf{op} \mid C \rangle$.

Proof. We observe that $\bigcirc_{\mathrm{grpd}}\, C \simeq \langle \flat \mid \bigcirc_{\mathrm{grpd}}\, C\rangle$ and likewise for $\langle \mathrm{op} \mid C\rangle$. Accordingly, we note that:

$$\begin{aligned}
\langle \flat \mid \langle \mathrm{op} \mid C\rangle \to \langle \flat \mid X\rangle\rangle &\simeq \langle \flat \mid \bigcirc_{\mathrm{grpd}} \langle \mathrm{op} \mid C\rangle \to \langle \flat \mid X\rangle\rangle \\
\langle \flat \mid C \to \langle \flat \mid X\rangle\rangle &\simeq \langle \flat \mid \bigcirc_{\mathrm{grpd}}\, C \to \langle \flat \mid X\rangle\rangle \\
\langle \flat \mid C \to \langle \flat \mid X\rangle\rangle &\simeq \langle \flat \mid \langle \mathrm{op} \mid C\rangle \to \langle \flat \mid X\rangle\rangle
\end{aligned}$$

Finally, the result follows from a simple Yoneda argument. □

The following is informed by the development of [48, Ch. 8].

Definition C.1 (Covariant equivalences). Fix $p :_\flat C \to A$ and $q :_\flat D \to A$ between categories A, C, D. Let $f :_\flat C \to D$ be a fibered map as follows:

$$\begin{array}{ccc}
C & \xrightarrow{\quad f \quad} & D \\
 & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle q} & \\
 & A &
\end{array}$$

We call f a *covariant equivalence* if for all families $X :_\flat A \to \mathcal{S}$ reindexing gives rise to an equivalence, i.e.,:

$$f^* : \left(\prod_{a:A} D_a \to X_a\right) \to \left(\prod_{a:A} C_a \to X_a\right)$$

Dually, f is called *contravariant equivalence* precomposition with respect to all contravariant families is an equivalence.

Lemma C.2. *Let f as below be a covariant equivalence with respect to p and q. Then, for any functor $r :_\flat B \to A$ it is also a covariant equivalence with respect to rp and rq:*

$$\begin{array}{ccc}
C & \xrightarrow{\quad f \quad} & D \\
 & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle q} & \\
 & A & \\
 & \downarrow{\scriptstyle r} & \\
 & B &
\end{array}$$

Proof. We get the following induced square:

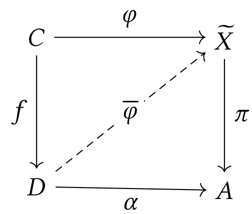
$$\begin{array}{ccc}
\left(\prod_{b:B} D_b \to X_b\right) & \xrightarrow[\simeq]{f^*} & \left(\prod_{b:B} C_b \to X_b\right) \\
\simeq\downarrow & & \downarrow\simeq \\
\left(\prod_{a:A} D_a \to X_{r(a)}\right) & \xrightarrow[f^*]{} & \left(\prod_{a:A} C_a \to X_{r(a)}\right)
\end{array}$$

The upper horizontal map is an equivalence by the preconditions. The goal is to show that the lower horizontal map is an equivalence, too. But this follows from 2-for-3 for equivalences. □

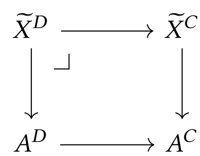
Lemma C.3 (Characterizations of (left) cofinality)**.** *Let $f :_\flat C \to D$ be a functor. Then the following are equivalent:*

(1) *f is left cofinal.*

(2) *Let $X :_\flat A \to \mathcal{S}$ be a family with associated left fibration $\pi :_\flat \widetilde{X} \to A$. Then any square of the following form has a filler $\overline{\varphi}$, uniquely up to homotopy:*



(3) *For any family $X :_\flat A \to \mathcal{S}$ the following square is a pullback:*



(4) *f is a covariant equivalence with respect to any $\alpha :_\flat D \to A$.*

The analogous characterization holds for contravariant equivalences and right cofinal functores.

Proof. Conditions (2) and (3) are readily seen to be equivalent by commuting $\prod$ and $\sum$. Condition (4) unfolds to the following: for any $X :_\flat A \to \mathcal{S}$ reindexing along f is an equivalence, namely

$$f^* : \left(\prod_{a:A} D_a \to X_a\right) \xrightarrow{\sim} \left(\prod_{a:A} \left(\sum_{d:D_a} C_{a,d}\right) \to X_a\right)$$

This, again, is readily seen to be equivalent to (2).

We turn to the implication (4) $\implies$ (1). But this is clear, since (1) says that f is a covariant equivalence with respect to itself and id_D.

For the converse direction (1) $\implies$ (4) we use the insight just made together with Lemma C.2. □

Proposition 5.16.* *$f :_\flat C \to D$ is left (right) cofinal if and only if for every covariant (contravariant) family $\pi :_\flat X \to Y$, f is left orthogonal to π, i.e.,* $\mathsf{isEquiv}(X^D \to X^C \times_{Y^C} Y^D)$.

Proof. Immediate by Lemma C.3 for the left cofinal case and by duality and Corollary 5.15 for the right cofinal case. □

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